DEPARTMENT OF ECONOMICS AND FINANCE

SCHOOL OF BUSINESS AND ECONOMICS

UNIVERSITY OF CANTERBURY

CHRISTCHURCH, NEW ZEALAND

Too Cold for Comfort: A Theoretical Analysis of Index-Based Insurance for Frost Damage to Crops

Amogh Prakasha Kumar
Laura Meriluoto
Richard Watt

WORKING PAPER

No. 2/2024

Department of Economics and Finance
UC Business School
University of Canterbury
Private Bag 4800, Christchurch
New Zealand

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Too Cold for Comfort: A Theoretical Analysis of Index-Based Insurance for Frost Damage to Crops

Amogh Prakasha Kumar[1]
Laura Meriluoto[1]
Richard Watt[1†]

February 2024

Abstract: In this article we reconsider the economic theory of index insurance. In the previous literature on this topic, the only choice variable that the insurance demander was permitted to choose was the amount of the payment to be received should the index be verified. However, it is clear that the index itself is an integral element in the contract, and it is an element about which the insurance demander has clear preferences. We analyse the optimal demand for index insurance under the assumption that the demander can choose both the payment to be received when the index is verified, and the index value itself. We find that there is always an interior solution to this problem. We also consider the market equilibrium under a variety of settings in regards the position of the insurer. The article is cast within the scope of insurance for frost damage to a crop, but all of the results are directly applicable to any peril for which a publicly observable signal exists.

Keywords: Index insurance, crop insurance, agricultural economics

JEL Classifications: D8, G22, Q1

[1] Department of Economics and Finance, University of Canterbury, NEW ZEALAND

† Corresponding author: Richard Watt. Email: richard.watt@canterbury.ac.nz

1 Introduction

Insurance exists so that risk-averse parties facing a risk can share the outcome of the peril in exchange for a premium payment. However, the insurer will naturally charge a premium that sufficiently compensates for not only the expected value of the insured loss (i.e. expected claims under the contract), but also covers for any claim auditing that would need to occur. Claim auditing costs are an example of transaction costs in an insurance arrangement. In some instances, the transaction costs can be large compared to the actual benefit of the contract, to either or even both of the two parties involved. A good example of such a situation is in the agricultural sector, above all in regards crop insurance. Farmers' plots may be quite large, and often quite remotely located. On top of that, there are clear issues of asymmetric information that need to be overcome before a regular indemnity contract would be possible. This has led to indemnity insurance for crop damage due to such perils as adverse weather being unpopular and not frequently used.

On the other hand, if there exists a publicly observable signal as to the damage that might have occurred on a farm, then that signal can be used as a proxy for the damage. That is the basis of index insurance arrangements, in which the indemnity payout is made conditional not upon the actual damage that occurs, but rather on the value of the signal (the value of the "index"). Such contracts are reasonably popular, at least in the agricultural sector of many less-developed countries.

There are many papers that look at index-based cover from an empirical perspective, but only Clarke (2016) analyses the theory.[1] However, Clarke only considers the question of optimal coverage given a value for the index, and not the question of preferences over the index itself. However, it is easy to see that the insured will have a preference for the index used, and indeed that there should be a relationship between the index chosen and the coverage chosen. Here we aim to fill that gap, using the case of insurance against frost damage. We consider the case of an insured who can optimally choose both the index and the level of coverage of their contract, that is, rather than facing a single-dimensional optimisation problem, the insured must solve a two-dimensional problem. We also consider the preferences of a risk-neutral insurer, and we characterise the equilibrium contracts for a perfectly competitive insurer.

The article first goes through some general theory related to the probabilities that are in question. Then we explore the expected utility of an index insurance product. In section 4 we provide a

[1] See also the currently unpublished work of Louaas and Picard (2022).

numerical simulation to show how the model will work. Section 5 gives some general propositions related to the optimal demand for index insurance, and section 6 looks at a few potential market outcomes. Section 7 concludes.

2 General theory

An index insurance contract provides a rather specific type of cover against a given peril. Instead of simply providing an indemnity that is functionally related to the value of the loss, the coverage offered for index insurance is functionally related to a variable that proxies the size of the loss - the "index". There are pros and cons of this sort of insurance as compared to normal indemnity coverage. Most importantly, index insurance is useful when it is costly to observe the actual size of a loss (including cases in which asymmetric information on the size of the loss exists), and there exists a publicly observable index outcome that can be monitored without cost. In such a case, the transaction costs of underwriting the risk are lowered (all claim auditing costs are avoided), and insurance can be offered at a lower premium. On the other hand, it is highly unlikely that there will exist an index that is a perfect proxy for the underlying loss, and so there is a non-zero probability of the loss happening although the index value doesn't reach the threshold needed for a payout. There is, of course, also the opposite possibility - the index reaches the defined threshold and so a payout happens, and yet no loss occurs. The disjoint between payout and occurance of loss is known as "basis risk".[2]

Index insurance has become relatively popular for agricultural insurance (see for example, Mukherjee et al. (2021), Fuchs and Wolff (2011), Carter et al. (2015), Miranda and Farrin (2012), Jensen and Barrett (2017), Sarris (2013), Matsura and Kurosaki (2019) and Johnson (2013)). In such cases, the main perils that need to be insured are often weather related - floods, high winds, droughts and temperature extremes. These sorts of events are certainly closely related to crop failure and damage (i.e., the weather outcomes are a signal for crop damage), and are publicly observable. At the same time, it is costly to observe individual farmers' actual damage values following a weather event, maybe due to asymmetric information, maybe due to the remote location of some farms. Therefore, index insurance has come to dominate many agricultural insurance markets. In the present article, we consider an index insurance arranagement that is related to temperature,

[2] In essence, index insurance is nothing more than a wager on the value of the index. The "insured" retains all of the actual underlying risk, but since the index wager would pay out when the loss is more probable, the former acts as a hedge against the latter.

with the case being that frost at the wrong time of the year causes irrepairable damage to fledgling crops. This is the case all over the world, for example, with wine growing. The spring buds that should develop into grapes to be harvested and used to produce wine can be damaged by late frosts, causing lower quantity and quality wine or even no wine at all. Since the incidence of frost is closely related to the minimum temperature that occurs, index insurance can be written using the minimum temperature as the index.[3] Throughout this article, we take the index signal to be binomial - either the threshold is reached, or it is not. Thus, the contract in question involves two variables; the value of the index threshold and the payout if the threshold is reached. It is assumed that there is no payout if the threshold is not reached. This is certainly the most common structure of index insurance in the real-world. Of course, a much more complex arrangement is possible, in which each value of the index would correspond to a value for the payout. That is, the payout would be a continuous function of the index value. We are not aware of any case of this type of contract having ever been used, and so we do not attempt to model it here.

Our model considers an index insurance product for frost damage to a crop. Naturally, the contract is restricted only to periods of time in which there is some non-zero probability of frost, and for which frost will indeed cause damage to the crop. For most countries the period of time in question would correspond to the spring, when crops are budding (fruiting flowers are forming) and frosts are still possible. We assume that if a frost does occur within the relevant period, then there will indeed be some amount of crop damage, and if a frost does not occur then there is no such damage. It is, of course, beside the point that there may exist other perils to the crop in question (e.g. droughts, floods, insect pests, fire hazards, etc.). We are only concerned with frost damage. We assume that the minimum temperature that is reached is a signal for the occurance of a frost (and therefore, for the occurance of crop damage from frost), such that the lower is the minimum temperature, the more likely it is that a frost occurs.

For all that follows, define t as *the minimum temperature that is reached during the period covered by an index insurance contract.*

We are interested in three probabilities, namely:

1. Given a value for the index i, the probability that $t \leq i$. Call this $p(i)$, and refer to it as the probability that the index is verified.

[3] We discuss the fact that temperature, even when it is very accurately measured at the actual location of the crop, is still only a signal of likelihood of frost. Also, it is almost impossible to expect that each farm would have its own weather station from which temperature measures are taken. Rather, the temperature signal will be taken from the closest public weather station, where again the temperature might be a little different from what is happening on the farm in question.

2. Given that the specified value for the index is reached (i.e. conditional on $t \leq i$ occuring), the probability that there is a frost. Call this $\Pi_z(i)$.

3. Given that the specified value of the index is not reached (i.e. conditional on $t > i$ occuring), the probability that there is a frost. Call this $\Pi_y(i)$.

Begin by assuming that there is a known probability density for t, denoted by $f(t)$, which is defined on $L \leq t \leq H$, where both $f(L)$ and $f(H)$ are positive and $f(t) = 0$ for all $t < L$ and for all $t > H$ (i.e. L is the lowest minimum temperature with positive probability, and H is the highest minimum temperature with positive probability). Then directly we have

$$p(i) = \int_L^i f(t)dt$$

Clearly $p(i)$ is monotone increasing, so a choice of i is exactly equivalent to a choice of a value for p. Using the definition of $p(i)$, we further define the density for the probability of the minimum temperature, conditional upon temperature being no larger than i, by

$$\text{for } t \leq i, \;\; z_i(t) = \frac{f(t)}{p(i)}$$

Clearly,

$$\int_L^i z_i(t)dt = 1$$

which tells us the obvious fact that, conditional upon $t \leq i$, the probability that indeed $t \leq i$ is equal to 1.

Likewise, we can define

$$\text{for } t > i, \;\; y_i(t) = \frac{f(t)}{1 - p(i)}$$

This is the density for the probability of the minimum temperature conditional upon temperature being no smaller than i. We now have

$$\int_i^H y_i(t)dt = 1$$

or, conditional upon $t \geq i$, the probability that indeed $t \geq i$ is equal to 1.

Second, consider the general probability that there is a frost at each temperature t, which we denote by $\pi(t)$. It is tempting to assume that there exists a known threshold $\widehat{t}$ such that $\pi(t) = 1$ for all $t \leq \widehat{t}$, and $\pi(t) = 0$ for all $t > \widehat{t}$. If this were to be the case and if, say, $\widehat{t} = 0$, then there would certainly be a frost if the minimum temperature is 0 or below, and there would never be a frost if the minimum temperature is above 0. But this cannot in reality be true, since if it were, simply by setting $i = \widehat{t}$ we would achieve an index contract with no basis risk at all (the only chance for a frost is if the index is reached, in which case the payout is given), and an indemnity contract would suffice. In short, temperature would be a perfect signal for frost. Therefore, it must rather be the case that $\pi(t)$ is asymptotic to 1 as t gets small (and in limit is equal to 1 at $t = L$), aysmptotic to 0 as t gets large (and in limit is equal to 0 at $t = H$), and strictly non-increasing for all t.

A typical looking case could be like that in Figure 1 (where zero degrees centigrade would fall somewhere in the steepest part of the graph).

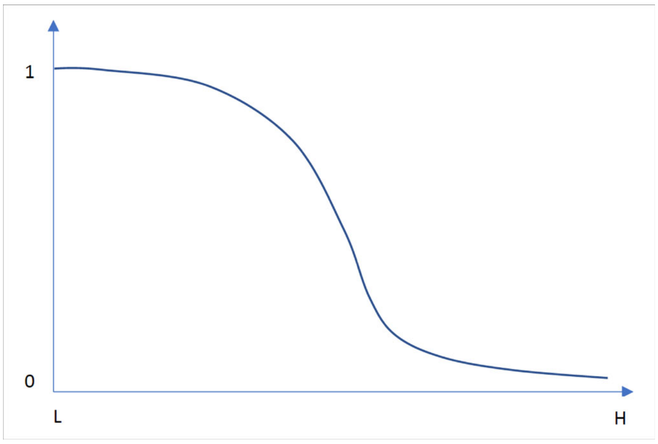

Figure 1: A typical graph of the probability of frost at each temperature, $\pi(t)$

Given that, then the probability of a frost occurring with a specific minimum temperature t is

$$\Pi(t) = \pi(t)f(t)$$

The unconditional probability of frost is

$$\Pi = \int_L^H \pi(t)f(t)dt$$

Notice that Π is independent of t (and of course independent of any chosen temperature index) but would of course be different for different insurable periods (as defined by different functions $f(t)$).

Now, we are in a position to define the probability of frost, conditional upon temperature being no larger than some given number i. It is

$$\begin{aligned}\Pi_z(i) &= \int_L^i \pi(t)z_i(t)dt \\ &= \int_L^i \frac{\pi(t)f(t)}{p(i)}dt \\ &= \frac{1}{p(i)}\int_L^i \pi(t)f(t)dt\end{aligned}$$

This is the probability of a frost occurring with any temperature $t \leq i$, that is, the conditional probability of frost, conditional upon $t \leq i$.

In exactly the same way, the probability of frost conditional upon temperature being no smaller than some given number i is

$$\begin{aligned}\Pi_y(i) &= \int_i^H \pi(t)y_i(t)dt \\ &= \int_i^H \frac{\pi(t)f(t)}{1-p(i)}dt \\ &= \frac{1}{1-p(i)}\int_i^H \pi(t)f(t)dt\end{aligned}$$

Notice that, cross-multiplying we have

$$\Pi_z(i)p(i) = \int_L^i \pi(t)f(t)dt \quad \text{and} \quad \Pi_y(i)\left(1-p(i)\right) = \int_i^H \pi(t)f(t)dt$$

Summing these two equations gives the expected result that the weighted average of the conditional probabilities is always equal to the unconditional probability of frost:

$$\Pi_z(i)p(i) + \Pi_y(i)\left(1-p(i)\right) = \int_L^i \pi(t)f(t)dt + \int_i^H \pi(t)f(t)dt = \Pi$$

The important thing to notice here is that a change in the index i generates endogenous changes in both conditional probabilities, but since the unconditional probability is unchanged, the way the conditional probabilities change is controlled to a certain degree.

In particular, Lemma 1 points out a series of important characteristics of our conditional probabilities.

Lemma 1 *(i) As* $p(i) \to 0$, $\Pi_y(i) \to \Pi$. *(ii) As* $p(i) \to 1$, $\Pi_z(i) \to \Pi$. *(iii) As* $p(i) \to 0$, $\Pi_z(i) \to 1$. *(iv) As* $p(i) \to 1$, $\Pi_y(i) \to 0$. *(v)* $\Pi_z(i)$ *and* $\Pi_z(i)$ *are both decreasing functions of* i. *(vi) For all values of* $p(i)$*, it must hold that* $\Pi_z(i) \geq \Pi \geq \Pi_y(i)$.

Proof. See Appendix 1. ■

Together, the characteristics mentioned in Lemma 1 imply that the temperature index is indeed a valid, but not perfect, signal of the incidence of frost. The general graphical representation of the two conditional probabilities as functions of the index (the actual shapes of these functions depends on the specific situation assumed, especially in regards $p(i)$) as shown in Figure 2:

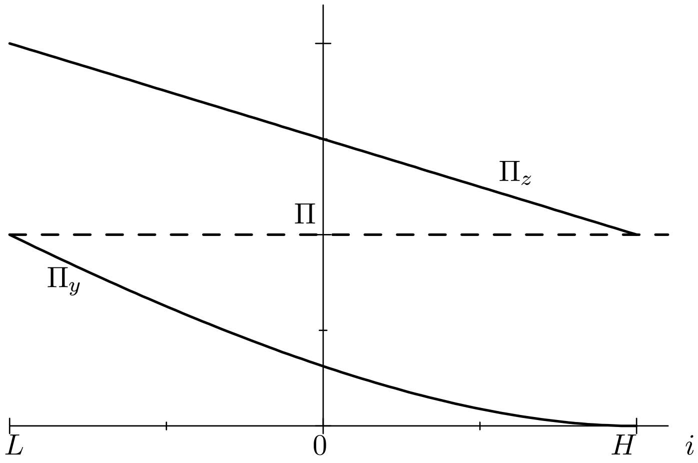

Figure 2: Conditional probabilities of frost

One final preliminary result will be handy later on:

Lemma 2 $\Pi'_z(i)|_{i=H} = f(H)\,(\pi(H) - \Pi) < 0.$

Proof. See Appendix 2. ■

3 Expected utility

The expected utility of an index insurance contract can be easily written down. In our model, it is a 4-state situation, depending on whether or not the index hits and whether or not there is frost. All we need to do is to make sure we have the right probabilities in place. Concretely, assume the following:

1. An index insurance contract is specified by a pair (i, Q) where i is a temperature index value, and Q is the payout for when temperature is less than or equal to the index.

2. The insured's utility function, $u(w)$, is strictly increasing and strictly concave.

3. The premium for a contract (i, Q) is $(1+\lambda)p(i)Q$, with $\lambda \geq 0$.

4. The insured crop is worth x_{nd} dollars if there is no frost (i.e. no damage), and it is worth x_d dollars, with $x_d < x_{nd}$, if there is frost (i.e. damage has occurred).

Then, the expected utility of a contract (i, Q) is:

$$EU(i,Q) = p(i)\left(\Pi_z(i)u\left(x_d - (1+\lambda)p(i)Q + Q\right) + (1-\Pi_z(i))\,u\left(x_{nd} - (1+\lambda)p(i)Q + Q\right)\right)$$

$$+(1-p(i))\left(\Pi_y(i)u\left(x_d - (1+\lambda)p(i)Q\right) + (1-\Pi_y(i))\,u\left(x_{nd} - (1+\lambda)p(i)Q\right)\right) \quad (1)$$

The farmer's objective is to choose (i, Q) so as to maximise $EU(i, Q)$, subject to $0 \leq Q \leq x_{nd} - x_d$ and $L \leq i \leq H$. The farmer's reservation utility can be found by setting $Q = 0$;

$$\begin{aligned} EU(i,Q)|_{Q=0} &= p(i)\left(\Pi_z(i)u\left(x_d\right) + (1-\Pi_z(i))\,u(x_{nd})\right) + (1-p(i))\left(\Pi_y(i)u\left(x_d\right) + (1-\Pi_y(i))\,u\left(x_{nd}\right)\right) \\ &= u\left(x_d\right)\left(p(i)\Pi_z(i) + (1-p(i))\Pi_y(i)\right) + u(x_{nd})\left(p(i)\left(1-\Pi_z(i)\right) + (1-p(i))\left(1-\Pi_y(i)\right)\right) \\ &= \Pi u\left(x_d\right) + (1-\Pi)u(x_{nd}) \end{aligned}$$

where we have used

$$\Pi_z(i)p(i) + \Pi_y(i)(1-p(i)) = \Pi$$

and

$$\begin{aligned} (1-p(i))(1-\Pi_y(i)) + p(i)\Pi_z(i) &= 1 - ((1-p(i))\Pi_y(i) + p(i)\Pi_z(i)) \\ &= 1 - \Pi \end{aligned}$$

The first-order conditions for an optimal contract are[4]

$$\frac{\partial EU(i,Q)}{\partial i} = 0 \text{ and } \frac{\partial EU(i,Q)}{\partial Q} = 0$$

We begin our analysis of optimal contracts with an illustrative simulation.

[4] The second-order condition for optimal Q holds by assumption of concavity of utility. The second-order condition for optimal i is far more complex. When neccesary, we will simply assume it holds as well.

4 A simulation

Since the equation for expected utility is a particularly complex function to derive, it is useful to inspect it by simulation and contructing graphs. To that end, assume the following:

1. The utility function is $u(w) = w^k$ with $0 < k < 1$ so as to capture risk aversion. Specifically, in all of the graphs below, $k = 0.5$.

2. The value of the crop with no frost is 60, and its value with frost is 40 (so the loss upon suffering frost is 20).

3. The contracted payout, Q, satisfies $0 \leq Q \leq 20$.

4. The density for minimum temperature over the contract period, $f(t)$, is uniform between $L = -4$ degrees and $H = +4$ degrees.

5. The probability of frost for each feasible minimum temperature is[5]

$$\pi(t) = \begin{cases} 1 & \text{if} \quad t \leq a \\ \frac{b-t}{b-a} & \text{if} \quad a < t < b \\ 0 & \text{if} \quad t \geq b \end{cases} \quad \text{with} \quad a = -1 \text{ and } b = 1$$

6. The index must be set between a and b.

The assumption on $f(t)$ gives $f(t) = \frac{1}{H-L}$, and $p(i) = \frac{i-L}{H-L}$. The graph of $\pi(t)$ is:

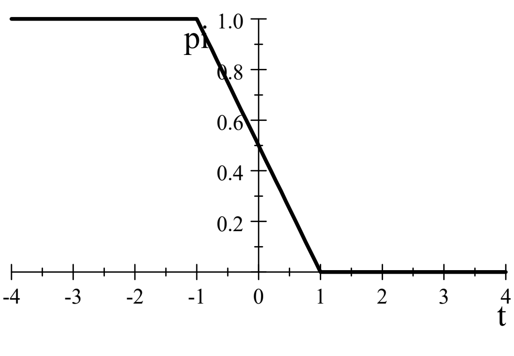

Figure 3: The graph of $\pi(t)$

[5]That is, t is distributed between L and H, there is a frost for sure if $L \leq t \leq a$, there is no frost for sure if $b \leq t \leq H$, and the probability of frost is a decreasing linear function for any t between a and b.

Using the (uniform) function for $f(t)$, the unconditional probability of frost with this function for $\pi(t)$ is:

$$\begin{aligned}\Pi &= \int_L^H \pi(t)f(t)dt \\ &= \left(\frac{1}{H-L}\right)\int_L^H \pi(t)dt \\ &= \left(\frac{1}{H-L}\right)\left((a-L)+\frac{1}{2}(b-a)\right)\end{aligned}$$

Using the parameter values $L=-4$, $H=4$, $a=-1$ and $b=1$, this is $\Pi = 0.5$.

The conditional probabilities need to be calculated as piecewise functions. They are the following (with $L=-4$, $H=4$, $a=-1$ and $b=1$):[6]

$$\Pi_z(i) = \begin{cases} 1 & \text{if } -4 \le i \le -1 \\ \left(\frac{1}{i+4}\right)\left(4-\frac{(1-i)^2}{4}\right) & \text{if } -1 \le i \le 1 \\ \left(\frac{4}{i+4}\right) & \text{if } 1 \le i \le 4 \end{cases}$$

and

$$\Pi_y(i) = \begin{cases} \left(\frac{-i}{4-i}\right) & \text{if } -4 \le i \le -1 \\ \left(\frac{1}{4-i}\right)\left(\frac{(1-i)^2}{4}\right) & \text{if } -1 \le i \le 1 \\ 0 & \text{if } 1 \le i \le 4 \end{cases}$$

The graphs of these two conditional probabilities are displayed in Figures 4 and 5 below:

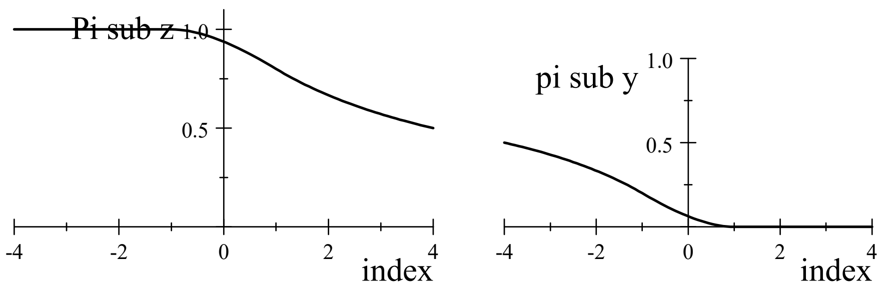

Figure 4: Probability of frost conditional upon the index being verified (left pane) or not (right pane)

[6] The calculations for these functions are in Appendix 3.

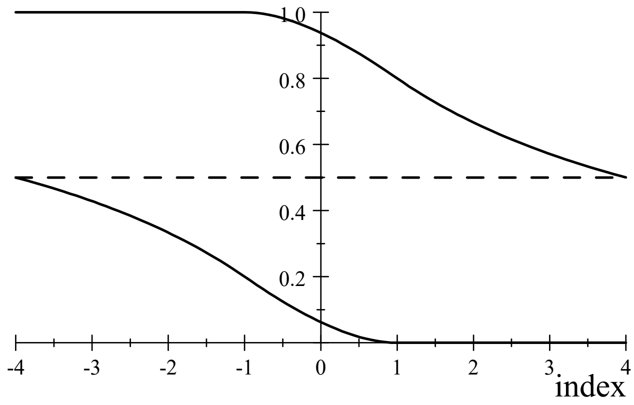

Figure 5: Both conditional probabilities of frost together

If the index is set at -1 then the probability of frost given the index is reached is 1, and if the index is set at 1, the probability of frost given the index is reached is about 0.8. With an index of 1, the probability of frost when the index is not reached is about 0.2, and if the index is set at 1, the probability of frost when the index is not reached is 0.

Using all of that information, and setting $\lambda = 0$ and $Q = 15$, the graph in Figure 6 is generated for expected utility as a function of the value of the index:

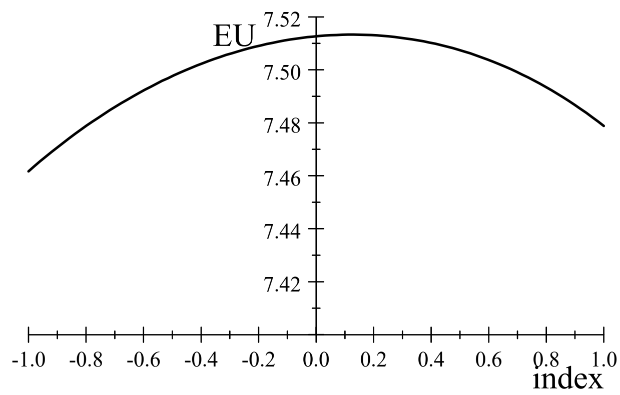

Figure 6: Expected utility for a given value of indemnity with $\lambda = 0$

We can see that expected utility is concave in the index, and is maximised at an index value of around 0.15.

Second, repeating the same exercise, but with $\lambda = 0.1$, the following graph is generated:

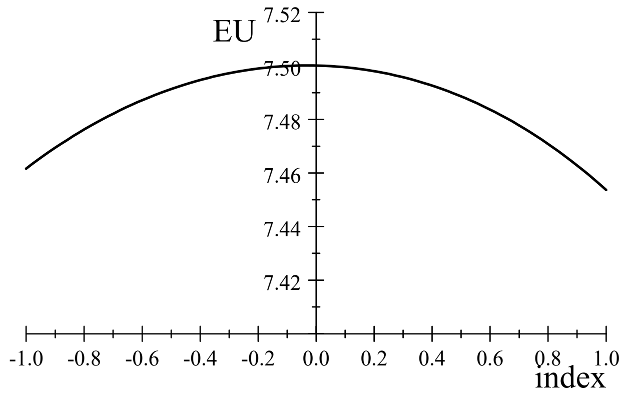

Figure 7: Expected utility for a given value of indemnity, with $\lambda = 0.1$

With this change from fair insurance to loaded insurance, the index value that the insured would prefer has gone down to something slightly below 0.

Now, set the index at $i = 0.2$, and consider the graph of expected utility as a function of Q. Figure 8 shows the resulting graph with $\lambda = 0$:

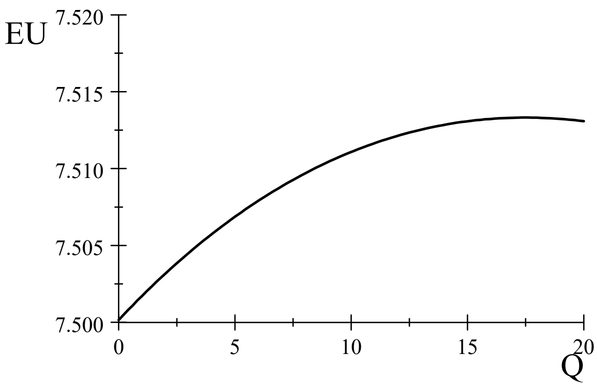

Figure 8: Expected utility with a given value of the index, with $\lambda = 0$

Given this index, the insured's optimal choice of Q is less than full coverage (a little below 18), even though premium is fair (in the sense that the premium is equal to the expected claim). This reflects the existence of basis risk (and it conforms with Clarke's (2016) proposition 1, even though the assumptions in the current model are somewhat different from those of Clarke).

Finally, keeping the index at 0.2, but increasing the loading factor up to $\lambda = 0.1$, gives the following graph:

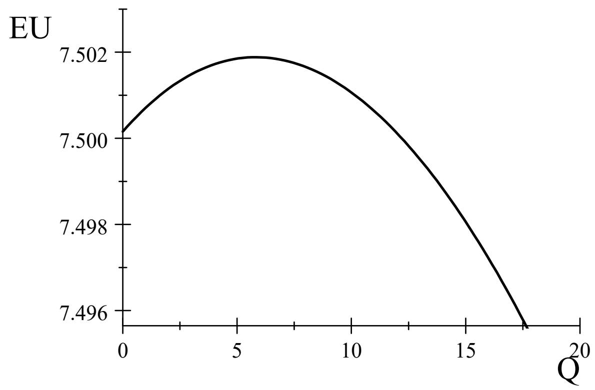

Figure 9: Expected utility for a given value of the index, with $\lambda = 0.1$

Here, it can be observed that the increase in the premium from a fair level to a loaded level has the effect of reducing the optimal Q significantly, down to about 6.

5 A few general results

The simulation of the previous section is illustrative of this problem, and it does highlight the need to include the index into the contract as a second decision variable along with the level of coverage. In this final section, we consider a few general results concerning the optimal contract.

Proposition 1 *For any given index value satisfying $L < i < H$, a positive amount of insurance is always purchased if the contract has a fair premium, $\lambda = 0$.*

Proof. The proof requires evaluating the first derivative of expected utility with respect to Q, and showing that this derivative takes a negative value at $Q = 0$. The expected utility of a contract at (i, Q) is given in equation (1) above, which for convenience is repeated here:

$$\begin{aligned} EU(i,Q) &= p(i)\left(\Pi_z(i)u\left(x_d - (1+\lambda)p(i)Q + Q\right) + (1-\Pi_z(i))\,u\left(x_{nd} - (1+\lambda)p(i)Q + Q\right)\right) \\ &\quad + (1-p(i))\left(\Pi_y(i)u\left(x_d - (1+\lambda)p(i)Q\right) + (1-\Pi_y(i))\,u\left(x_{nd} - (1+\lambda)p(i)Q\right)\right) \end{aligned}$$

The derivative of this with respect to Q is:

$$\begin{aligned}\frac{\partial EU}{\partial Q} &= p(i)\Pi_z(i)u'\left(x_d-(1+\lambda)p(i)Q+Q\right)\left(1-(1+\lambda)p(i)\right)\\ &+p(i)\left(1-\Pi_z(i)\right)u'\left(x_{nd}-(1+\lambda)p(i)Q+Q\right)\left(1-(1+\lambda)p(i)\right)\\ &+(1-p(i))\Pi_y(i)u'\left(x_d-(1+\lambda)p(i)Q\right)\left(-(1+\lambda)p(i)\right)\\ &+(1-p(i))\left(1-\Pi_y(i)\right)u'\left(x_{nd}-(1+\lambda)p(i)Q\right)\left(-(1+\lambda)p(i)\right)\end{aligned}$$

At $\lambda = 0$, this becomes

$$\begin{aligned}\frac{\partial EU}{\partial Q} &= p(i)\Pi_z(i)u'\left(x_d+Q(1-p(i)\right)\left(1-p(i)\right)\\ &+p(i)\left(1-\Pi_z(i)\right)u'\left(x_{nd}+Q(1-p(i)\right)\left(1-p(i)\right)\\ &+(1-p(i))\left(\Pi_y(i)u'\left(x_d-p(i)Q\right)\right)\left(-p(i)\right)\\ &+(1-p(i))\left(1-\Pi_y(i)\right)u'\left(x_{nd}-p(i)Q\right)\left(-p(i)\right)\end{aligned}$$

Divide through by $p(i)\left(1-p(i)\right)$ to get

$$sign\frac{\partial EU}{\partial Q} = sign[\Pi_z(i)u'\left(x_d+Q(1-p(i)\right)+\left(1-\Pi_z(i)\right)u'\left(x_{nd}+Q(1-p(i)\right)$$
$$-\left(\left(\Pi_y(i)u'\left(x_d-p(i)Q\right)\right)+\left(1-\Pi_y(i)\right)u'\left(x_{nd}-p(i)Q\right)\right)]$$

Now, substitute into this the value $Q = 0$,

$$sign\left.\frac{\partial EU}{\partial Q}\right|_{Q=0} = sign\left[\Pi_z(i)u'\left(x_d\right)+\left(1-\Pi_z(i)\right)u'\left(x_{nd}\right)-\left(\left(\Pi_y(i)u'\left(x_d\right)\right)+\left(1-\Pi_y(i)\right)u'\left(x_{nd}\right)\right)\right]$$

$$\begin{aligned}sign\left.\frac{\partial EU}{\partial Q}\right|_{Q=0} &= sign\left[u'\left(x_d\right)\left(\Pi_z(i)-\Pi_y(i)\right)+u'\left(x_{nd}\right)\left(1-\Pi_z(i)-1+\Pi_y(i)\right)\right]\\ sign\left.\frac{\partial EU}{\partial Q}\right|_{Q=0} &= sign\left[u'\left(x_d\right)\left(\Pi_z(i)-\Pi_y(i)\right)+u'\left(x_{nd}\right)\left(\Pi_y(i)-\Pi_z(i)\right)\right]\\ sign\left.\frac{\partial EU}{\partial Q}\right|_{Q=0} &= sign\left[\left(\Pi_z(i)-\Pi_y(i)\right)\left(u'\left(x_d\right)-u'\left(x_{nd}\right)\right)\right]\end{aligned}$$

By Lemma 1 $\Pi_z(i) > \Pi_y(i)$, and since $x_d < x_{nd}$ and the utility function is concave $u'\left(x_d\right) > u'\left(x_{nd}\right)$. Therefore, on any contract with $\lambda = 0$, marginal utility at $Q = 0$ is strictly positive, indicating that there exist values of $Q > 0$ that are preferred to no coverage. ■

Proposition 2 *Assume $\lambda = 0$ and $L < i < H$. Full coverage is not optimal.*

Proof. In the proof of the previous proposition, it was shown that with $\lambda = 0$ and $L < i < H$ we have

$$sign\frac{\partial EU}{\partial Q} = sign[\Pi_z(i)u'\left(x_d + Q(1-p(i)\right) + \left(1-\Pi_z(i)\right)u'\left(x_{nd} + Q(1-p(i)\right)$$
$$-\left(\left(\Pi_y(i)u'\left(x_d - p(i)Q\right)\right) + \left(1-\Pi_y(i)\right)u'\left(x_{nd} - p(i)Q\right)\right)]$$

This sign is non-positive (indicating that the farmer should reduce his contracted coverage) if

$$\begin{aligned}&\Pi_z(i)u'\left(x_d + Q(1-p(i)\right) + \left(1-\Pi_z(i)\right)u'\left(x_{nd} + Q(1-p(i)\right)\\ \leq\ &\Pi_y(i)u'\left(x_d - p(i)Q\right) + \left(1-\Pi_y(i)\right)u'\left(x_{nd} - p(i)Q\right)\end{aligned}$$

Substitute into this the value $Q = x_{nd} - x_d$ (i.e. full coverage):

$$\Pi_z(i)u'\left(x_d + \left(x_{nd} - x_d\right)\left(1-p(i)\right)\right) + \left(1-\Pi_z(i)\right)u'\left(x_{nd} + \left(x_{nd} - x_d\right)\left(1-p(i)\right)\right)$$
$$\leq \Pi_y(i)u'\left(x_d - p(i)\left(x_{nd} - x_d\right)\right) + \left(1-\Pi_y(i)\right)u'\left(x_{nd} - p(i)\left(x_{nd} - x_d\right)\right) \qquad (2)$$

Notice that both sides of this are simple convex combinations. The left-hand side is therefore smaller than $u'\left(x_d + \left(x_{nd} - x_d\right)\left(1-p(i)\right)\right)$, and the right-hand side is larger than $u'\left(x_{nd} - p(i)\left(x_{nd} - x_d\right)\right)$. However, notice that

$$\begin{aligned} x_d + \left(x_{nd} - x_d\right)\left(1-p(i)\right) &= x_d(1-(1-p(i)) + x_{nd}(1-p(i)) \\ &= x_d p(i) + x_{nd} - x_{nd}p(i) \\ &= x_{nd} - p(i)(x_{nd} - x_d) \end{aligned}$$

If we define $x_{nd} - p(i)(x_{nd} - x_d) \equiv k$, then we have that the left-hand side of (2) is smaller than $u'(k)$ and the right-hand side is larger than $u'(k)$. Therefore, at full coverage $\left.\frac{\partial EU}{\partial Q}\right|_{Q=x_{nd}-x_d} < 0$.
■

Propositions 1 and 2 mimic the results reported by Clarke's (2016) Proposition 1, where it is implicitly assumed that $L < i < H$. The most important result is that with an index insurance product that is priced at an actuarially fair premium, the insured demands partial coverage. The reason is of course that even at full coverage, index insurance leaves the insured with a certain amount of risk - the so-called basis risk in the contract. Full coverage is not really an option under an index contract.

Propositions 1 and 2 specifically deal with the baseline case of actuarially fair insurance. In that case, there will always be an interior solution, so long as the index is not set at an extreme. When we consider an actuarially loaded premium, $\lambda > 0$, then it becomes possible that optimal demand falls to 0. As is well known from the general indemnity insurance demand problem, the comparative statics of insurance loading are ambiguous (there is both an income and a substitution effect), however, by continuity, in our problem positive values of λ that are sufficiently small (i.e. close to 0) will still generate an internal solution. Of course, as λ increases, it will eventually reach a value such that optimal demand falls to 0. The limit value of λ such that insurance demand falls to 0 is that value at which optimal expected utility equals reservation utility. To that end, take the threshold value of the loading factor to be $\widehat{\lambda}$.

Proposition 3 *Assume the index is set at some an index that satisfies $L < i < H$. Then if the loading factor is set at*

$$\widehat{\lambda}(i) = \frac{\Pi_z(i)u'(x_d) + (1 - \Pi_z(i))\, u'(x_{nd})}{\Pi u'(x_d) + (1 - \Pi)\, u'(x_{nd})} - 1$$

or higher, the optimal demand for index insurance is 0.

Proof. At $\widehat{\lambda}$, by assumption the optimal demand for coverage is $Q^* = 0$. So, substitute those two values into the formula above for marginal expected utility:

$$\begin{aligned}\left.\frac{\partial EU}{\partial Q}\right|_{Q^*=0} &= p(i)\Pi_z(i)u'(x_d)\left(1 - (1+\widehat{\lambda})p(i)\right) + p(i)\,(1 - \Pi_z(i))\, u'(x_{nd})\left(1 - (1+\widehat{\lambda})p(i)\right) \\ &\quad + (1 - p(i))\Pi_y(i)u'(x_d)\left(-(1+\widehat{\lambda})p(i)\right) + (1 - p(i))\,(1 - \Pi_y(i))\, u'(x_{nd})\left(-(1+\widehat{\lambda})p(i)\right)\end{aligned}$$

Collecting common terms, this is

$$\begin{aligned}&u'(x_d)\left(p(i)\Pi_z(i)\left(1 - (1+\widehat{\lambda})p(i)\right) - (1 - p(i))\Pi_y(i)(1+\widehat{\lambda})p(i)\right) \\ &+ u'(x_{nd})\left(p(i)\,(1 - \Pi_z(i))\left(1 - (1+\widehat{\lambda})p(i)\right) - (1 - p(i))\,(1 - \Pi_y(i))\,(1+\widehat{\lambda})p(i)\right)\end{aligned}$$

Notice that there is a common factor of $p(i) > 0$, which we can ignore since in the end our expression must equal 0 for $Q^* = 0$ to be the optimal response to $\widehat{\lambda}$. Thus we have

$$\begin{aligned} & u'(x_d)\left(\Pi_z(i)\left(1-(1+\widehat{\lambda})p(i)\right)-(1-p(i))\Pi_y(i)(1+\widehat{\lambda})\right) \\ & +u'(x_{nd})\left((1-\Pi_z(i))\left(1-(1+\widehat{\lambda})p(i)\right)-(1-p(i))\left(1-\Pi_y(i)\right)(1+\widehat{\lambda})\right) \\ = \; & u'(x_d)\left(\Pi_z(i)-(1+\widehat{\lambda})\left(p(i)\Pi_z(i)+(1-p(i))\Pi_y(i)\right)\right) \\ & +u'(x_{nd})\left((1-\Pi_z(i))-(1+\widehat{\lambda})\left(p(i)\left(1-\Pi_z(i)\right)+(1-p(i))\left(1-\Pi_y(i)\right)\right)\right) \end{aligned}$$

We have already seen above that

$$p(i)\Pi_z(i)+(1-p(i))\Pi_y(i)=\Pi$$

so our expression is

$$u'(x_d)\left(\Pi_z(i)-(1+\widehat{\lambda})\Pi\right)+u'(x_{nd})\left((1-\Pi_z(i))-(1+\widehat{\lambda})(1-\Pi)\right)=0$$

that is

$$\Pi_z(i)u'(x_d)+(1-\Pi_z(i))\,u'(x_{nd})=(1+\widehat{\lambda})\left(\Pi u'(x_d)+(1-\Pi)\,u'(x_{nd})\right)$$

Cross multiplying, we find that the limit value of the loading factor satisfies

$$1+\widehat{\lambda}=\frac{\Pi_z(i)u'(x_d)+(1-\Pi_z(i))\,u'(x_{nd})}{\Pi u'(x_d)+(1-\Pi)\,u'(x_{nd})}$$

Since $\Pi_z(i) > \Pi$, and $u'(x_d) > u'(x_{nd})$, the fraction on the right-hand side is a strictly finite number greater than 1. Therefore, there exists a finite value $\widehat{\lambda} > 0$ such that optimal demand for our index insurance product goes to 0. ■

Corollary 1 $\widehat{\lambda}(i)$ *is a decreasing function of* i*, with*

$$\widehat{\lambda}(L)=\frac{(1-\Pi)\left(u'(x_d)-u'(x_{nd})\right)}{\Pi u'(x_d)+(1-\Pi)\,u'(x_{nd})} \quad \textit{and} \quad \widehat{\lambda}(H)=0$$

Proof. We only need to differentiate $\widehat{\lambda}(i)=\frac{\Pi_z(i)u'(x_d)+(1-\Pi_z(i))u'(x_{nd})}{\Pi u'(x_d)+(1-\Pi)u'(x_{nd})}-1$. Clearly, the first derivative is

$$\widehat{\lambda}'(i)=\frac{\Pi_z'(i)\left(u'(x_d)-u'(x_{nd})\right)}{\Pi u'(x_d)+(1-\Pi)\,u'(x_{nd})}$$

Since $u'(x_d) > u'(x_{nd})$, and $\Pi_z'(i) < 0$, we get $\widehat{\lambda}'(i) > 0$. Furthermore,

$$\begin{aligned}
\widehat{\lambda}(L) &= \frac{\Pi_z(L)u'(x_d) + (1-\Pi_z(L))\,u'(x_{nd})}{\Pi u'(x_d) + (1-\Pi)\,u'(x_{nd})} - 1 \\
&= \frac{u'(x_d)}{\Pi u'(x_d) + (1-\Pi)\,u'(x_{nd})} - 1 \\
&= \frac{u'(x_d) - (\Pi u'(x_d) + (1-\Pi)\,u'(x_{nd}))}{\Pi u'(x_d) + (1-\Pi)\,u'(x_{nd})} \\
&= \frac{(1-\Pi)\,(u'(x_d) - u'(x_{nd}))}{\Pi u'(x_d) + (1-\Pi)\,u'(x_{nd})}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\lambda}(H) &= \frac{\Pi_z(H)u'(x_d) + (1-\Pi_z(H))\,u'(x_{nd})}{\Pi u'(x_d) + (1-\Pi)\,u'(x_{nd})} - 1 \\
&= \frac{\Pi u'(x_d) + (1-\Pi)\,u'(x_{nd})}{\Pi u'(x_d) + (1-\Pi)\,u'(x_{nd})} - 1 \\
&= 0
\end{aligned}$$

So $\widehat{\lambda}(i)$ runs from the value $\frac{(1-\Pi)(u'(x_d)-u'(x_{nd}))}{\Pi u'(x_d)+(1-\Pi)u'(x_{nd})}$ at $i = L$ down to the value 0 at $i = H$. ■

Corollary 2 *Assuming $L < i < H$, there will be a strictly interior solution for the insured's choice of coverage for all λ such that $0 \leq \lambda < \widehat{\lambda}$.*

We now move on to consider the optimal choice of the index, something that was not done by Clarke (2016). We remind the reader that expected utility is:

$$\begin{aligned}
EU(i,Q) &= p(i)\,(\Pi_z(i)u\,(x_d - (1+\lambda)p(i)Q + Q) + (1-\Pi_z(i))\,u\,(x_{nd} - (1+\lambda)p(i)Q + Q)) \\
&\quad + (1-p(i))\,(\Pi_y(i)u\,(x_d - (1+\lambda)p(i)Q) + (1-\Pi_y(i))\,u\,(x_{nd} - (1+\lambda)p(i)Q))
\end{aligned}$$

To begin with, consider the two extreme choices of the index, $i = H$ and $i = L$.

Proposition 4 *If the chosen index is either $i = H$ or $i = L$, the insured ends up with reservation utility. Thus, there is nothing to be gained by selecting either of those two options.*

Proof. Consider first $i = H$, which corresponds to $p = 1$, $\Pi_z = \Pi$ and $\Pi_y = 0$. Thus, we get

$$\begin{aligned}
EU(i,Q)|_{i=H} &= \Pi u\,(x_d - (1+\lambda)Q + Q) + (1-\Pi)\,u\,(x_{nd} - (1+\lambda)Q + Q) \\
&= \Pi u\,(x_d - \lambda Q) + (1-\Pi)\,u\,(x_{nd} - \lambda Q)
\end{aligned}$$

Clearly, this is less than reservation utility unless $Q = 0$, in which case the insured gets exactly reservation utility. Therefore, there is nothing at all to be gained by setting $i = H$. Second, consider the option $i = L$, for which $p = 0$, $\Pi_z = 1$ and $\Pi_y = \Pi$. This gives expected utility of

$$EU(i, Q)|_{i=L} = \Pi u(x_d) + (1 - \Pi) u(x_{nd})$$

That is, independently of the value of coverage that is chosen, expected utility is equal to its reservation value. Again, there is nothing to be gained by setting $i = L$. ■

Proposition 4 points clearly towards the solution always involving an intermediate value of the index, that is i would be set such that $0 < p(i) < 1$.

Proposition 5 *The optimal value of the index, i^*, satisfies $i^* < H$.*

Proof. See Appendix 4. ■

We can now state the following;

Proposition 6 *The optimal value of the index, i^*, must be internal, that is $L < i^* < H$.*

Proof. Proposition 5 proves the second inequality. But, from Proposition 4, choosing an index value at either end of the feasible range gives the same value for expected utility, namely reservation expected utility. Therefore, since (by Proposition 5) there exist index values smaller than H that are preferred to H, those index choices must also dominate choosing the value L. ■

Corrollary 2 and Proposition 4 show that, subject to the loading factor being less than the threshold value identified in Proposition 3, the optimal solution to the index insurance demand problem must be internal, that is $0 < Q^* < x_{nd} - x_d$ and $L < i^* < H$. This result does not preclude there being more than one optimal point. However, it is true that the optimum is unique if expected utility is concave in Q and in i. The first is simple enough to prove, but the second is arduous. In what follows, we assume that there is a unique optimal solution to the problem, and then from our propositions above, that point is interior. In common economic parlance, there exists an internal "bliss point" for the insurance demand problem.

6 Market equilibrium

It is useful to consider the sorts of equilibria that can occur in this game, where the insurer is risk-neutral and the insured is risk-averse. It is easiest to consider this problem in terms of the indifference curves of each party in (p, Q) space. Recall that since $p(i)$ is monotone increasing, a choice of i directly translates into a choice of p, so we can think about our indifference curves as functions of the level of coverage and the probability of a claim, given i. The fact that the insurance demander will always choose an interior value for i directly tells us that we only need to consider values of p that satisfy $0 < p < 1$. We begin with the insurer.

Given a contract at (p, Q), and given a loading factor of λ between 0 and $\widehat{\lambda}$, and assuming that the insurer only faces the actuarial costs of the contract (i.e. no fixed costs), the insurer earns expected profit of

$$EB(i, Q) = (1 + \lambda)pQ - pQ = \lambda pQ$$

Assuming $\lambda > 0$ and $Q > 0$, an indifference curve for expected profit equal to a constant $\overline{EB}$ can directly be seen to be

$$p = \frac{\overline{EB}}{\lambda Q}$$

This is a negatively sloped convex curve in (p, Q) space. Moreover, the points for which $EB = 0$ are those that coincide with the axes of the graph, while any other (positive) value of expected profit has an indifference curve that passes through the space in question. Curves further from the origin indicate higher levels of expected profit, that is, the insurer's preference direction is towards the north-east. A set of such curves is shown below for $\lambda > 0$. The dashed lines indicate the feasible set, $0 \leq p \leq 1$ and $0 \leq Q \leq Q_f$, where Q_f indicates full coverage. Since any curve that is to the left of the vertical axis and above the horizontal axis represents positive expected profit, the entire feasible set of contracts constitutes the contracts that are acceptable to the insurer.

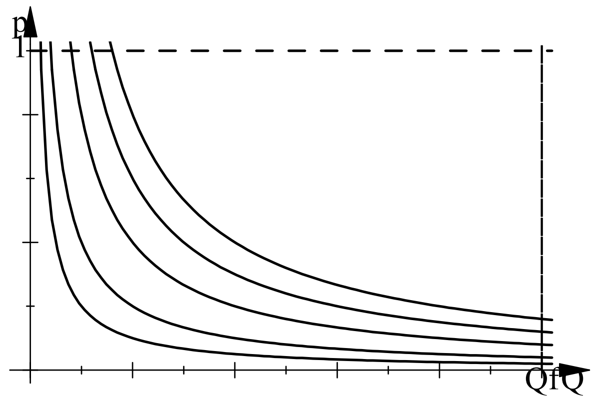

Figure 10: Indifference curves for the insurer

Second, consider the insurance demander. We know that there is an internal optimum for the demand problem, and so the indifference curves are concentric oval-shaped curves centred on the internal bliss point (which we here assume is unique), and with curves further from the bliss point indicating lower levels of expected utility. We know that the reservation indifference curve coincides with the two axes, but it will have a concave upper boundary that passes through the space of feasible contracts. A set of such indifference curves, with the reservation utility curve in darker font, is sketched below with the bliss point located at B. Any contract point inside the reservation utility curve is an acceptable contract for the demander.[7]

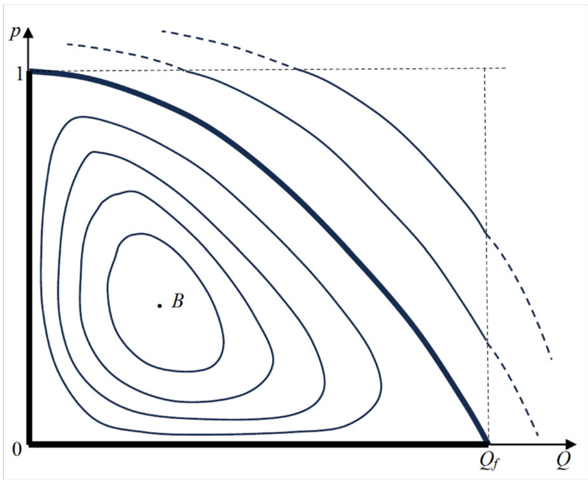

Figure 11: Indifference curves for insurance demander

To find the market equilibrium, all we need to do is to begin by plotting the insurance demander's reservation utility curve, and their bliss point. Then, we locate the insurer's indifference curve that is tangent to the demander's reservation curve, and the insurer's indifference curve that passes through the demander's bliss point. The first of these (point M below) is the solution when the insurer is a monopolist, and the second point (the demander's bliss point, point B below) is the

[7] In this figure, the upper boundary of the reservation curve passes through the point $(p, Q) = (1, 0)$ and through the point $(p, Q) = (0, Q_f)$. While those two points are certainly on the reservation curve, it can turn out that the upper-boundary of that curve can reach the limit value of p to the right of the point $(1, 0)$ and/or it can reach the limit value of Q to the left of $(0, Q_f)$.

solution when the insurer is perfectly competitive.

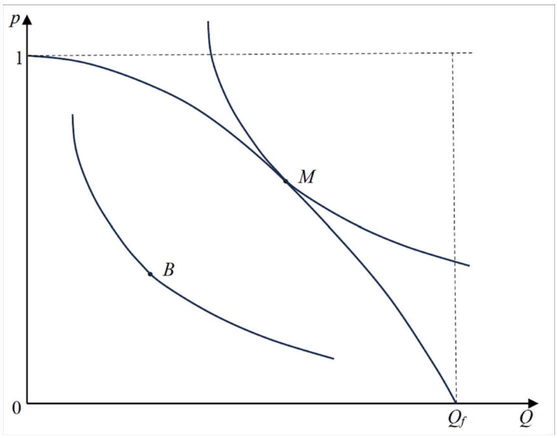

Figure 12: Market equilibrium points for perfect competition and monopoly

Interestingly, then, at the equilibrium contract, a perfectly competitive insurer will earn a strictly positive expected profit.

Finally, it is also worthwhile to consider the dependence of the optimal level of indemnity Q, upon the level of the index chosen, i. Here, as above, we will show this using the probability p instead of the index itself. Consider then the next Figure. Say the index is chosen such that we have p_2. Then the insurance demander will optimally select an indemnity of Q_2. Similarly, Q_1 will be chosen when we have p_1, and of course Q^* will correspond to p^*. The curve joining the points A, B and C, which is labelled $p(Q)$, is the (inverse) curve for the function linking the index to optimal indemnity. It must pass through the parts of each indifference curve that have zero slope, where normally there will be two such points for each indifference contour.

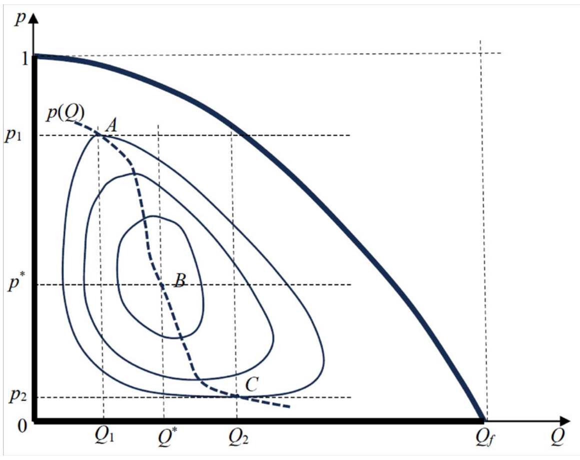

Figure 13: Optimal index insurance demand as the index changes

We can think of $p(Q)$ as a sort of demand function for this market. Imagine that the game were to be that the insurer moves first with an offer of indemnity insurance at a given level of the index, or equivalently, a given number for p, and the demander then takes that number and selects her optimal demand, $Q(p)$, which is given by the inverse of $p(Q)$. What number p would a profit maximising insurer select? The answer is simply that the insurer would choose the point on $p(Q)$ that is tangent to their indifference curve (see the point (p_E, Q_E) in Figure 14).

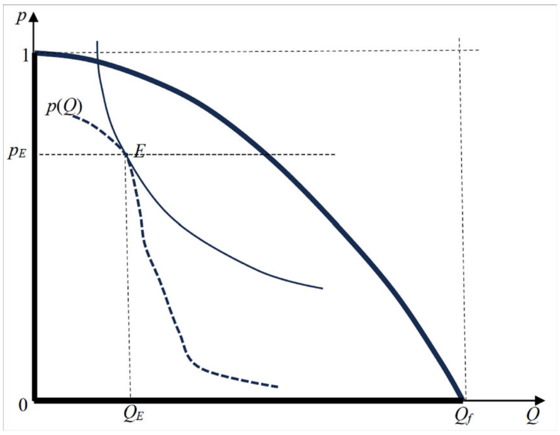

Figure 14: Equilibrium when a profit maximising insurer chooses p and the insurance demander chooses Q

7 Conclusions

In this article, we have considered optimal demand for an index insurance product. Our model is motivated by the case of frost insurance for agricultural crops, a market in which index insurance is not at all uncommon. In contrast to the existing theoretical literature on this topic, we allow the insurance demander to determine both the level of indemnity to be received if the index is verified, as well as the value of the index itself. We have confirmed the results from the existing literature in regards the optimal demand for indemnity - the solution is interior, and is limited by a threshold value of the insurance loading factor. However, we also show that the optimal index value is also interior, and of course will depend on the demander's utility function (risk aversion), and the specific probability functions that are of issue. Different situations will result in different levels of the index being chosen. The fact that the solution (i^*, Q^*) is strictly interior to the feasible contract set implies that the demander has a bliss-point.

Our analysis of the optimal demand for index insurance leads to a very simple theory of market equilibrium. The solution is a standard tangency between indifference curves when the insurer is a monopolist (expected profit maximising), and it is at the bliss-point when the insurer is perfectly competitive. In the latter case, the insurer will still earn positive expected profits in the equilibrium, in contrast with the case of regular indemnity insurance markets.

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Appendix 1: Proof of Lemma 1

Characteristics (i) and (ii) are easily proved by finding the relevant limits. Since

$$\Pi_z(i) = \frac{\int_L^i \pi(t)f(t)dt}{\int_L^i f(t)dt}$$

and since $p(i) \to 1$ is equivalent to $i \to H$, we have

$$\lim_{i \to H} \frac{\int_L^i \pi(t)f(t)dt}{\int_L^i f(t)dt} = \frac{\int_L^H \pi(t)f(t)dt}{\int_L^H f(t)dt} = \frac{\Pi}{1} = \Pi$$

Similarly, since $p(i) \to 0$ is equivalent to $i \to L$, and given that

$$\Pi_y(i) = \frac{\int_i^H \pi(t)f(t)dt}{\int_i^H f(t)dt}$$

we get

$$\lim_{i \to L} \frac{\int_i^H \pi(t)f(t)dt}{\int_i^H f(t)dt} = \frac{\int_L^H \pi(t)f(t)dt}{\int_L^H f(t)dt} = \frac{\Pi}{1} = \Pi$$

Characteristics (iii) and (iv) can both be proved using L'Hopital's rule (recalling that $p'(i) = f(i)$). Specifically, consider

$$\lim_{p(i) \to 0} \Pi_z(i) = \lim_{i \to L} \frac{\int_L^i \pi(t)f(t)dt}{\int_L^i f(t)dt}$$

As $i \to L$, both the numerator and the denominator go to 0. So, using L'Hopital's rule,

$$\begin{aligned} \lim_{i \to L} \frac{\int_L^i \pi(t)f(t)dt}{\int_L^i f(t)dt} &= \left. \frac{\pi(i)f(i)}{f(i)} \right|_{i=L} \\ &= \pi(L) \\ &= 1 \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{p(i) \to 1} \Pi_y(i) &= \lim_{i \to H} \frac{\int_i^H \pi(t)f(t)dt}{\int_i^H f(t)dt} \\ &= \left. \frac{-\pi(i)f(i)}{-f(i)} \right|_{i=H} \\ &= \pi(H) \\ &= 0 \end{aligned}$$

Characteristic (v), the fact that the two conditional probabilities are both strictly decreasing functions, can be proved by differentiation. Take $\Pi_z(i)$. Since

$$\Pi_z(i) = \frac{\int_L^i \pi(t)f(t)dt}{\int_L^i f(t)dt}$$

it's first-derivative is

$$\begin{aligned}
\Pi_z'(i) &= \frac{\pi(i)f(i)\int_L^i f(t)dt - \int_L^i \pi(t)f(t)dt \times f(i)}{\left(\int_L^i f(t)dt\right)^2} \\
&= \frac{f(i)}{\left(\int_L^i f(t)dt\right)^2}\left(\pi(i)\int_L^i f(t)dt - \int_L^i \pi(t)f(t)dt\right) \\
&= \frac{f(i)}{\left(\int_L^i f(t)dt\right)^2}\left(\int_L^i \pi(i)f(t)dt - \int_L^i \pi(t)f(t)dt\right) \\
&= \frac{f(i)}{\left(\int_L^i f(t)dt\right)^2}\left(\int_L^i \left(\pi(i) - \pi(t)\right) f(t)dt\right)
\end{aligned}$$

However, since $\pi(i)$ is a strictly decreasing function, $\pi(i) \leq \pi(t)$ for all $t \leq i$ (with strict inequality for $t < i$). Thus $\int_L^i \left(\pi(i) - \pi(t)\right) f(t)dt < 0$, and so $\Pi_z'(i) < 0$. Likewise, given that

$$\Pi_y(i) = \frac{\int_i^H \pi(t)f(t)dt}{\int_i^H f(t)dt}$$

it's first derivative is

$$\begin{aligned}
\Pi_y'(i) &= \frac{-\pi(i)f(i)\int_i^H f(t)dt - \int_i^H \pi(t)f(t)dt \times (-f(i))}{\left(\int_i^H f(t)dt\right)^2} \\
&= \frac{-\pi(i)f(i)\int_i^H f(t)dt + \int_i^H \pi(t)f(t)dt \times f(i)}{\left(\int_i^H f(t)dt\right)^2} \\
&= \frac{f(i)}{\left(\int_i^H f(t)dt\right)^2}\left(\int_i^H \pi(t)f(t)dt - \pi(i)\int_i^H f(t)dt\right) \\
&= \frac{f(i)}{\left(\int_i^H f(t)dt\right)^2}\left(\int_i^H \left(\pi(t) - \pi(i)\right) f(t)dt\right)
\end{aligned}$$

However, since $\pi(t) \leq \pi(i)$ on $t \geq i$ (again with strict inequality for $t > i$), we get $\int_i^H \left(\pi(t) - \pi(i)\right) f(t)dt < 0$, and correspondingly $\Pi_y'(i) < 0$.[8]

Finally, characteristic (vi) is implied by the first 5.

[8] Both of these slope results rely on $\pi(t)$, the probability of frost given temperature, being strictly decreasing. Of course, it is likely that there is some minimum temperature below which the probability of frost is always 1 (i.e. $\pi(t)$ is flat on that zone), as well as some maximum temperature above which the probability of frost is always 0 (again, $\pi(t)$ is flat). On those zones, the conditional probabilities will also be flat.

Appendix 2: Proof of Lemma 2

In the previous appendix, it is shown that

$$\Pi_z'(i) = \frac{f(i)}{\left(\int_L^i f(t)dt\right)^2}\left(\int_L^i (\pi(i)-\pi(t))\, f(t)dt\right)$$

Therefore,

$$\begin{aligned}\Pi_z'(H) &= \frac{f(H)}{\left(\int_L^H f(t)dt\right)^2}\left(\int_L^H (\pi(H)-\pi(t))\, f(t)dt\right)\\ &= f(H)\left(\pi(H)\int_L^H f(t)dt - \int_L^H \pi(t)f(t)dt\right)\\ &= f(H)\,(\pi(H)-\Pi)\end{aligned}$$

This is negative since $f(H) > 0$, and since $\pi(t)$ is decreasing, we have $\pi(H) < \pi(t)$ for all $t < H$. Therefore

$$\int_L^H \pi(H)f(t)dt < \int_L^H \pi(t)f(t)dt$$

or

$$\pi(H) < \Pi.$$

Appendix 3: Probability calculations for the simulation

In this appendix, we present an outline of the calculations for the conditional probabilities in the simulation. The density for the minimum temperature, $f(t)$ is assumed to be uniform, between L and H. Thus, $f(t) = \frac{1}{H-L}$, and $p(i) = \int_L^i f(t)dt = \frac{i-L}{H-L}$. Clearly, then, $1-p(i) = \frac{H-i}{H-L}$. The probability of frost for each feasible temperature is assumed to be

$$\pi(t) = \begin{cases} 1 & \text{if} \quad t \le a \\ \frac{b-t}{b-a} & \text{if} \quad a < t < b \\ 0 & \text{if} \quad t \ge b \end{cases}$$

Finally, the parameter values are $L = -4$, $H = 4$, $a = -1$ and $b = 1$.

Take first the zone $-4 \leq i \leq -1$. On that zone, we have $\pi(t) = 1$, and so

$$\begin{aligned}
\Pi_z(i) &= \frac{1}{p(i)} \int_L^i \pi(t) f(t) dt \\
&= \frac{1}{p(i)} \int_L^i f(t) dt \\
&= \frac{p(i)}{p(i)} \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
\Pi_y(i) &= \frac{1}{1 - p(i)} \int_i^H \pi(t) f(t) dt \\
&= \frac{H - L}{H - i} \frac{1}{H - L} \int_i^H \pi(t) dt \\
&= \frac{1}{H - i} \left(\int_i^a \pi(t) dt + \int_a^b \pi(t) dt + \int_b^H \pi(t) dt \right)
\end{aligned}$$

Using the concrete formulas for $\pi(t)$ on each zone, this is

$$\begin{aligned}
\Pi_y(i) &= \frac{1}{H - i} \left(\int_i^a 1 dt + \int_a^b \frac{b - t}{b - a} dt \right) \\
&= \frac{1}{H - i} \left(a - i + \frac{1}{2}(b - a) \right) \\
&= \frac{1}{H - i} \left(\frac{a + b}{2} - i \right)
\end{aligned}$$

With the parameter values, this is

$$\Pi_y(i) = \frac{-i}{4 - i}$$

Second, take the interval $-1 \leq i \leq 1$:

$$\begin{aligned}
\Pi_z(i) &= \frac{1}{p(i)} \int_L^i \pi(t) f(t) dt \\
&= \frac{H - L}{i - L} \frac{1}{H - L} \int_L^i \pi(t) dt \\
&= \frac{1}{i - L} \left(\int_L^a \pi(t) dt + \int_a^i \pi(t) dt \right)
\end{aligned}$$

Using the relevant expressions for $\pi(t)$, this is

$$\begin{aligned}
\Pi_z(i) &= \frac{1}{i-L}\left(\int_L^a 1dt + \int_a^i \frac{b-t}{b-a}dt\right) \\
&= \frac{1}{i-L}\left(a - L - \frac{1}{2(b-a)} \times (b-t)^2\Big|_a^i\right) \\
&= \frac{1}{i-L}\left(a - L - \frac{(b-i)^2 - (b-a)^2}{2(b-a)}\right) \\
&= \frac{1}{i-L}\left(a - L - \frac{(b-i)^2}{2(b-a)} + \frac{(b-a)^2}{2(b-a)}\right) \\
&= \frac{1}{i-L}\left(a - L - \frac{(b-i)^2}{2(b-a)} + \frac{(b-a)}{2}\right) \\
&= \frac{1}{i-L}\left(\frac{a+b}{2} - L - \frac{(b-i)^2}{2(b-a)}\right)
\end{aligned}$$

Using our parameter values, this becomes

$$\Pi_z(i) = \frac{1}{i+4}\left(4 - \frac{(1-i)^2}{4}\right)$$

Second, $\Pi_y(i)$ for $-1 \leq i \leq 1$ is calculated as follows.

$$\begin{aligned}
\Pi_y(i) &= \frac{1}{1-p(i)} \int_i^H \pi(t)f(t)dt \\
&= \frac{1}{H-i}\left(\int_i^b \pi(t)dt + \int_b^H \pi(t)dt\right) \\
&= \frac{1}{H-i}\int_i^b \frac{b-t}{b-a}dt \\
&= \frac{1}{H-i} \times \frac{1}{b-a} \times \int_i^b (b-t)dt \\
&= \frac{1}{H-i} \times \frac{1}{b-a} \times \left(-\frac{1}{2}\,(b-t)^2\Big|_i^b\right) \\
&= \frac{1}{H-i} \times \frac{-1}{2(b-a)} \times \left(-(b-i)^2\right) \\
&= \left(\frac{1}{H-i}\right)\left(\frac{(b-i)^2}{2(b-a)}\right)
\end{aligned}$$

The final zone to consider is $1 \leq i \leq 4$. On that zone, given that $\pi(t) = 0$, we have

$$\Pi_y(i) = \frac{1}{1-p(i)} \int_i^H \pi(t)f(t)dt = 0$$

Lastly, $\Pi_y(i)$ for $1 \leq i \leq 4$ is calculated as follows.

$$\begin{aligned}
\Pi_z(i) &= \frac{1}{p(i)} \int_L^i \pi(t) f(t) dt \\
&= \frac{H-L}{i-L} \frac{1}{H-L} \int_L^i \pi(t) dt \\
&= \frac{1}{i-L} \left(\int_L^a \pi(t) dt + \int_a^b \pi(t) dt + + \int_b^i \pi(t) dt \right) \\
&= \frac{1}{i-L} \left(\int_L^a 1 dt + \int_a^b \frac{b-t}{b-a} dt + 0 \right) \\
&= \frac{1}{i-L} \left((a-L) + \frac{1}{2}(b-a) \right) \\
&= \frac{1}{i-L} \left(\frac{a+b}{2} - L \right)
\end{aligned}$$

With the parameter values, this is

$$\Pi_z(i) = \frac{4}{i+4}$$

Appendix 4: Proof of proposition 5

Define $y_d(i) = x_d - (1+\lambda)p(i)Q$ and $y_{nd}(i) = x_{nd} - (1+\lambda)p(i)Q$, so $y_d'(i) = y_{nd}'(i) \equiv y'(i) = -(1+\lambda)p'(i)Q$. Expected utility is then

$$\begin{aligned}
EU(i,Q) &= p(i) \left(\Pi_z(i) u\left(y_d(i) + Q\right) + \left(1 - \Pi_z(i)\right) u\left(y_{nd}(i) + Q\right) \right) \\
&\quad + (1 - p(i)) \left(\Pi_y(i) u\left(y_d(i)\right) + \left(1 - \Pi_y(i)\right) u\left(y_{nd}(i)\right) \right)
\end{aligned}$$

Derive this with respect to i:

$$\begin{aligned}
\frac{\partial EU(i,Q)}{\partial i} &= p'(i) \left(\Pi_z(i) u\left(y_d(i) + Q\right) + \left(1 - \Pi_z(i)\right) u\left(y_{nd}(i) + Q\right) \right) \\
&\quad + p(i) \left(\Pi_z'(i) u\left(y_d(i) + Q\right) + \Pi_z(i) u'\left(y_d(i) + Q\right) y_d'(i) \right) \\
&\quad + p(i) \left(-\Pi_z'(i) u\left(y_{nd}(i) + Q\right) \right) + \left(1 - \Pi_z(i)\right) u'\left(y_{nd}(i) + Q\right) y_{nd}'(i) \\
&\quad - p'(i) \left(\Pi_y(i) u\left(y_d(i)\right) + \left(1 - \Pi_y(i)\right) u\left(y_{nd}(i)\right) \right) \\
&\quad + (1 - p(i)) \left(\Pi_y'(i) u\left(y_d(i)\right) + \Pi_y(i) u'\left(y_d(i)\right) y_d'(i) \right) \\
&\quad + (1 - p(i)) \left(-\Pi_y'(i) u\left(y_{nd}(i)\right) + \left(1 - \Pi_y(i)\right) u'\left(y_{nd}(i)\right) y_{nd}'(i) \right)
\end{aligned}$$

$$\begin{aligned}\frac{\partial EU(i,Q)}{\partial i} &= p'(i)\left(\Pi_z(i)u\left(y_d(i)+Q\right)+\left(1-\Pi_z(i)\right)u\left(y_{nd}(i)+Q\right)\right)\\ &-p'(i)\left(\Pi_y(i)u\left(y_d(i)\right)+\left(1-\Pi_y(i)\right)u\left(y_{nd}(i)\right)\right)\\ &+p(i)\Pi_z'(i)\left(u\left(y_d(i)+Q\right)-u\left(y_{nd}(i)+Q\right)\right)\\ &+p(i)\left(\Pi_z(i)u'\left(y_d(i)+Q\right)+\left(1-\Pi_z(i)\right)u'\left(y_{nd}(i)+Q\right)\right)y'(i)\\ &+(1-p(i))\Pi_y'(i)\left(u\left(y_d(i)\right)-u\left(y_{nd}(i)\right)\right)\\ &+(1-p(i))\left(\Pi_y(i)u'\left(y_d(i)\right)+\left(1-\Pi_y(i)\right)u'\left(y_{nd}(i)\right)\right)y'(i)\end{aligned}$$

Collect common terms, we see that $\frac{\partial EU(i,Q)}{\partial i}$ is equal to;

$$\begin{aligned}&p'(i)\left(\Pi_z(i)u\left(y_d(i)+Q\right)+\left(1-\Pi_z(i)\right)u\left(y_{nd}(i)+Q\right)-\left(\Pi_y(i)u\left(y_d(i)\right)+\left(1-\Pi_y(i)\right)u\left(y_{nd}(i)\right)\right)\right)\\ &+p(i)\left(\Pi_z'(i)\left(u\left(y_d(i)+Q\right)-u\left(y_{nd}(i)+Q\right)\right)+\left(\Pi_z(i)u'\left(y_d(i)+Q\right)+\left(1-\Pi_z(i)\right)u'\left(y_{nd}(i)+Q\right)\right)y'(i)\right)\\ &+(1-p(i))\left(\Pi_y'(i)\left(u\left(y_d(i)\right)-u\left(y_{nd}(i)\right)\right)+\left(\Pi_y(i)u'\left(y_d(i)\right)+\left(1-\Pi_y(i)\right)u'\left(y_{nd}(i)\right)\right)y'(i)\right)\end{aligned}$$

Now, as i approaches H, we have $p(H)=1$, $p'(H)=f(H)$, and (from Lemma 1) $\Pi_z=\Pi$ and $\Pi_y=0$. Finally, from Lemma 2, we have $\lim_{i\to H}\Pi_z'(i)=f(H)(\pi(H)-\Pi)$. Substitute all of this into the expression above for $\frac{\partial EU(i,Q)}{\partial i}$;

$$\begin{aligned}\left.\frac{\partial EU(i,Q)}{\partial i}\right|_{i=H} &= f(H)\left(\Pi u\left(x_d-\lambda Q\right)+\left(1-\Pi\right)u\left(x_{nd}-\lambda Q\right)-u\left(x_{nd}-\lambda Q\right)\right)\\ &+f(H)(\pi(H)-\Pi)\left(u\left(x_d-\lambda Q\right)-u\left(x_{nd}-\lambda Q\right)\right)\\ &-\left(\Pi u'\left(x_d-\lambda Q\right)+\left(1-\Pi\right)u'\left(x_{nd}-\lambda Q\right)\right)(1+\lambda)f(H)Q\end{aligned}$$

Cancel common terms on the first line;

$$\begin{aligned}\left.\frac{\partial EU(i,Q)}{\partial i}\right|_{i=H} &= f(H)\Pi\left(u\left(x_d-\lambda Q\right)-u\left(x_{nd}-\lambda Q\right)\right)\\ &+f(H)(\pi(H)-\Pi)\left(u\left(x_d-\lambda Q\right)-u\left(x_{nd}-\lambda Q\right)\right)\\ &-\left(\Pi u'\left(x_d-\lambda Q\right)+\left(1-\Pi\right)u'\left(x_{nd}-\lambda Q\right)\right)(1+\lambda)f(H)Q\end{aligned}$$

Combine the first and second lines;

$$\begin{aligned}\left.\frac{\partial EU(i,Q)}{\partial i}\right|_{i=H} &= \left(u\left(x_d-\lambda Q\right)-u\left(x_{nd}-\lambda Q\right)\right)\left(f(H)\Pi+f(H)(\pi(H)-\Pi)\right)\\ &-\left(\Pi u'\left(x_d-\lambda Q\right)+\left(1-\Pi\right)u'\left(x_{nd}-\lambda Q\right)\right)(1+\lambda)f(H)Q\end{aligned}$$

Simplify the first line;

$$\left.\frac{\partial EU(i,Q)}{\partial i}\right|_{i=H} = \left(u\left(x_d - \lambda Q\right) - u\left(x_{nd} - \lambda Q\right)\right) f(H)\pi(H) - \big(\Pi u'\left(x_d - \lambda Q\right) + (1-\Pi)\, u'\left(x_{nd} - \lambda Q\right)\big)(1+\lambda) f(H) Q$$

The first line is negative since $u\left(x_d - \lambda Q\right) < u\left(x_{nd} - \lambda Q\right)$ and $f(H)\pi(H) > 0$, and the second line is negative since marginal utility is positive. Therefore, $\left.\frac{\partial EU(i,Q)}{\partial i}\right|_{i=H} < 0$, which indicates that there exist choices for the index smaller than H such that expected utility is higher than at H. The optimal choice of the index can never be H.