DEPARTMENT OF ECONOMICS AND FINANCE

SCHOOL OF BUSINESS AND ECONOMICS

UNIVERSITY OF CANTERBURY

CHRISTCHURCH, NEW ZEALAND

Overlooked Results on the Competitive Firm under Output Price Risk: Alternative Sufficient Conditions For Downward Sloping Factor Demand Curves

Richard Watt

WORKING PAPER

No. 11/2020

Department of Economics and Finance
School of Business
University of Canterbury
Private Bag 4800, Christchurch
New Zealand

WORKING PAPER No. 11/2020

Overlooked Results on the Competitive Firm under Output Price Risk: Alternative Sufficient Conditions For Downward Sloping Factor Demand Curves

Richard Watt[1†]

July 2020

Abstract: The result that there are no Giffen input factors for a perfectly competitive firm under full certainty is one of the kingpins of basic undergraduate microeconomics. When the firm knows perfectly the price at which output can be sold, then the demand for each factor is a decreasing function of the unit wage of that factor. However, does that basic result on the slope of input demand functions still hold under more general conditions? Specifically, if the price at which the output is to be sold is stochastic, is it still true that the demand of a given firm for its inputs is unconditionally decreasing in the wage that is paid to obtain that input? The existing literature on this topic shows that factor demand curves are downward sloping whenever the firm's utility function for profits satisfies decreasing absolute risk aversion. In the present paper, using the simplest possible model, I show that there are alternative sufficient conditions, related to the concept of relative risk aversion, with no requirement to assume DARA.

Keywords: price risk, competitive firms, labor demand

JEL Classifications: D2, D8

[1] Department of Economics and Finance, University of Canterbury, NEW ZEALAND

†Corresponding author: Richard Watt, email: richard.watt@canterbury.ac.nz

1 Introduction

The result that there are no Giffen input factors for a perfectly competitive firm under full certainty is one of the king-pins of basic undergraduate microeconomics. When the firm knows perfectly the price at which output can be sold, then the demand for each factor is a decreasing function of the unit wage of that factor. However, does that basic result on the slope of input demand functions still hold under more general conditions? Specifically, if the price at which the output is to be sold is stochastic, is it still true that the demand of a given firm for its inputs is unconditionally decreasing in the wage that is paid to obtain that input? The existing literature on this topic shows that factor demand curves are downward sloping whenever the firm's utility function for profits satisfies decreasing absolute risk aversion. In the present paper, using the simplest possible model, I show that there are alternative sufficient conditions, related to the concept of relative risk aversion, with no requirement to assume DARA.

2 Literature

The early literature on the decisions of a competitive firm under price uncertainty dates back to the early 1970s (see, for perhaps the most relevant seminal papers, Baron 1970, Sandmo 1971, and Leyland, 1972). However, the principal focus of those papers was the comparison of firm decisions regarding output under price risk with those same decisions under full certainty. Curiously, very little attention was ever paid to the actual shape of the input demand functions.

Baron (1970) does not attempt any analysis of a firm's demand for factors under price uncertainty, but rather his analysis centres upon the question of how optimal output changes with risk aversion, given a risky output price. Baron shows that, for a given risk on price, optimal output is smaller the larger is the risk aversion of the firm. Analysing a similar problem as Baron, but in a somewhat simpler and more conventional fashion, Sandmo (1971) shows (among other results) that optimal output with a risky price is smaller than optimal output under certainty with price equal to the mean price in the risky scenario. Clearly, from both Baron (1970) and Sandmo (1971), given positive marginal product of labor it follows that, if labor is the only input, its optimal demand with a risky price is smaller than when price is certain (see Hartman 1975). But this says nothing about the shape of the demand curve for labor as a function of the wage paid.

Rather than looking specifically at random price, Leyland (1972) considers a random demand for the firm's output. Thus, although Leyland's work applies more to firms with a certain amount of market power, special cases of the results are still of interest to our present question. In any case, Leyland again only considers the effect of demand uncertainty upon the choice of output (for firms that either set quantity and face an uncertain price, or which set price and face an uncertain quantity). Leyland does not consider at any point how demand uncertainty affects the slope of the firm's demand for inputs.

Hartman (1976) does look specifically at factor demand under price uncertainty, using a two-input one-output model, but assuming that one of the factors, specifically labor, can be chosen after price uncertainty has been resolved. In such a model, the demand for labor is taken essentially under certainty, and the standard certainty conclusion, that labor demand is decreasing in the wage, applies.

More relevant to the present paper, Batra and Ullah (1974) specifically looked at the input demand of a competitive firm under price uncertainty. In a model with two inputs, Batra and Ullah extend the analysis of Sandmo (1971) by showing that an increase in price risk reduces the firm's optimal choice of quantity to produce. Since the model has two inputs, many of Batra and Ullah's results revolve around the shape of the isoquant curves associated with the production function, $f(L, K)$. Specifically, it is required that both marginal products are positive, and that f be quasi-concave, so that the isoquants are convex. As far as input demand is concerned, Batra and Ullah show an increase in price uncertainty leads to lower demands for inputs, due to lowering the firm's choice of output,[1] which is a result that is implied by, but not mentioned by, Sandmo (1971), and which in the Batra and Ullah article relies upon positive marginal products of the factors together with decreasing absolute risk aversion (DARA). Specifically in relation to the slope of the input demand functions expressed as functions of the wage for labor, Batra and Ullah note that the question depends on the sign of the cross-derivative of the production function, f_{KL}. In their conclusion, they state "... with well-behaved production functions (so that $f_{ii} < 0$ and $f_{KL} > 0$) and with our hypotheses concerning the firm's risk attitude, a rise in the price of any input induces the firm to reduce the demand for that input and lower its output. However, the firm's response toward the demand for the other input is indeterminate." The hypothesis concerning the firm's risk attitude that is mentioned is that the firm is risk averse, with decreasing absolute risk aversion. Therefore, Batra and Ullah show that, under DARA, and a concave production function, each input is decreasing in its own wage.

One can see that, starting with the relatively simple model of Sandmo (1971), the literature on the topic of price uncertainty with a competitive firm has tended to search for more complex modelling assumptions, thereby moving away from the simplest models. Here, I take the opposite tack, namely, I take a step back to the very basic model of a perfectly

[1] However, see Hartman (1975) who points out an error in the logic of this result.

competitive firm, producing a single output, x, using a single input, L, which I refer to as labor. My interest is in the shape of the firm's optimal demand for labor, $L^*(w)$, in particular the sign of it's slope, $L^{*\prime}(w)$. My simple model allows the analysis to concentrate completely on the issue of the shape of the demand for the input, when the only change over the basic undergraduate certainty case is the introduction of risk on the firm's output price, without any interference from other complexities in the production process.

3 Optimal labor demand function with a risky output price

Assume a firm that produces an output, x, using only labor, L, as an input. The production function, $x = f(L)$ is assumed to be strictly increasing and concave; $f'(L) > 0 > f''(L)$, and $f(0) = 0$. The wage that is paid for hiring labor is assumed to be exogenously determined at an amount w. The price at which the firm's output is sold, $\widetilde{p}$, is a random variable, with known density. Once price is determined, the firm's monetary profit is $pf(L) - wL$, and the firm must choose L, prior to knowing the outcome of price, to maximize expected utlity of profit, $Eu(pf(L) - wL)$. It is assumed that the firm's utility function is everywhere strictly increasing and strictly concave, $u' > 0 > u''$. We also assume that monetary profit is always non-negative, which essentially establishes a lower bound on the price distribution. The maximisation problem is

$$\max_{L} \; Eu(\widetilde{p}f(L) - wL)$$

The only constraint is that $L \geq 0$, although we will ignore that in all that follows, simply assuming that it is satisfied.

The first-order condition for an optimal labor choice is

$$Eu'(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) = 0 \tag{1}$$

The second-order condition is satisfied by the assumptions of concavity of utility and of the production function.

Notice first that, if there were no risk on the output price, so that $\widetilde{p} = p$ with certainty, then the optimal demand for labor, L_0^*, is found from

$$u'(pf(L_0^*) - wL_0^*)(pf'(L_0^*) - w) = 0$$

from which, given positive marginal utility, directly we see that with a certain output price the optimal labor demand satisfies $pf'(L_0^*) = w$. Thus, in this case, implicit differentiation reveals that

$$L_0^{*\prime}(w) = \frac{1}{pf''(L_0^*)} < 0$$

This is the standard certainty result that the input factor always has a negatively sloped demand curve.

When the output price is risky, and using the first-order condition given by (1), it becomes more difficult to evaluate the effect of a change in the wage upon the optimal labor demand. From the implicit function theorem, we find that

$$\text{sign } L^{*\prime}(w) = \text{ sign } \frac{d}{dw} Eu'(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w)$$

However, the derivative of the first-order condition with respect to the wage, which is what we need to put a sign on, is

$$-L^* Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) - Eu'(\widetilde{p}f(L^*) - wL^*) \tag{2}$$

The second term, $-Eu'(\widetilde{p}f(L^*) - wL^*)$ is strictly negative due to positive marginal utility, but the first term, $-L^* Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w)$ has ambiguous sign, since $pf'(L^*) - w$ will be negative for some values of p and positive for others. Indeed, we already know that if the price were to be certain, at some feasible value p, then the optimal solution is to set labor at an amount such that $pf'(L) - w = 0$. Holding the L in question constant, clearly at higher values of price, we would have $pf'(L) - w > 0$ while at lower values $pf'(L) - w < 0$. All we know in the case of a risky price is that the optimal demand for labor must satisfy $E(\widetilde{p}f'(L^*) - w) > 0$. This can be seen by rewriting the first-order condition as

$$Eu'(\widetilde{p}f(L^*) - wL^*)E(\widetilde{p}f'(L^*) - w) + cov(u'(\widetilde{p}f(L^*) - wL^*), \widetilde{p}f'(L^*) - w) = 0$$

An increase in p will increase profit, and thus decrease marginal utility of profit. At the same time, an increase in p will increase marginal revenue product $(pf'(L^*))$. Therefore, we have the result that $cov(u'(\widetilde{p}f(L^*) - wL^*), \widetilde{p}f'(L^*) - w) < 0$, and consequently, at the optimal labor demand it must happen that $Eu'(\widetilde{p}f(L^*) - wL^*)E(\widetilde{p}f'(L^*) - w) > 0$. Since we know for sure that $Eu'(\widetilde{p}f(L^*) - wL^*) > 0$, it must then be the case that $E(\widetilde{p}f'(L^*) - w) > 0$.

Notice that the fact that $E(\widetilde{p}f'(L^*) - w) > 0$ when the output price is risky tells us that optimal labor demand under risk is lower than optimal labor demand when the output price is certain but with the same mean, and also that optimal output is lower under risk (a result that was emphasised in the early literature). The equation $E(\widetilde{p}f'(L^*) - w) > 0$ can be written as $f'(L^*) > \frac{w}{E\widetilde{p}}$. If the output price were certain at an amount $p = E\widetilde{p}$, then optimal labor demand satisfies $pf'(L_0^*) = w$, that is, $f'(L_0^*) = \frac{w}{p} = \frac{w}{E\widetilde{p}}$. Therefore, we can directly see that $f'(L^*) > f'(L_0^*)$, which, since the production function is increasing and concave, tells us that a pure risk reduces optimal output, $f(L^*) < f(L_0^*)$, and it also reduces optimal demand for labor, $L^* < L_0^*$.

If we are to establish the slope of the demand for labor in this model, we need to put a sign on the expression (2) above. Specifically, we would like to find conditions such that (2) is negative. There is more than one way to do this, and I now go through three alternative options.

3.1 Option 1

We want to establish sufficient conditions for the following equation to hold;

$$-L^* Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) - Eu'(\widetilde{p}f(L^*) - wL^*) < 0$$

Clearly, it would be sufficient if the first term were non-positive, that is, if

$$-L^* Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) \leq 0$$

Dividing by $-L^*$, we see that what we require is

$$Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) \geq 0$$

This is, essentially, the same inequality used by Batra and Ullah (1974) to show that factor demands are decreasing in their own price under DARA (see Batra and Ullah, 1974, pp. 546-7).

To prove the result, notice that since $u''(z) = -A(z)u'(z)$, where $A(z)$ is the Arrow-Pratt measure of absolute risk aversion given wealth of z, we can write the inequality required as

$$E\left(-A(\widetilde{p}f(L^*) - wL^*)\right) u'(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) \geq 0$$

Take $\widehat{p}$ such that $\widehat{p}f'(L^*) - w = 0$. In that way, since $f'(L^*) > 0$ we have $pf'(L^*) - w < 0$ if $p < \widehat{p}$, and $pf'(L^*) - w > 0$ if $p > \widehat{p}$. Denote profits by $\pi(p) = pf(L) - wL$, so that $\pi'(p) > 0$, that is, profit π is increasing in p. Finally, define $\widehat{\pi} = \pi(\widehat{p})$. If we assume DARA, then

$$\begin{aligned} A(\pi) &> A(\widehat{\pi}) \text{ for all } p < \widehat{p} \\ A(\pi) &< A(\widehat{\pi}) \text{ for all } p > \widehat{p} \end{aligned}$$

Multiply these inequalities by $-u'(\pi)(pf'(L^*)-w)$, which is positive when $p<\widehat{p}$ and negative when $p>\widehat{p}$:

$$\begin{aligned} -A(\pi)u'(\pi)(pf'(L^*)-w) &> -A(\widehat{\pi})u'(\pi)(pf'(L^*)-w) \text{ for all } p<\widehat{p} \\ -A(\pi)U'(\pi)(pf'(L^*)-w) &> -A(\widehat{\pi})U'(\pi)(pf'(L^*)-w) \text{ for all } p>\widehat{p} \end{aligned}$$

Combining these, we get

$$-A(\pi)u'(\pi)(pf'(L^*)-w) > -A(\widehat{\pi})u'(\pi)(pf'(L^*)-w) \text{ for all } p$$

with equality if $p=\widehat{p}$. Taking expectations, we end up with

$$-EA(\pi)u'(\pi)(pf'(L^*)-w) > -A(\widehat{\pi})Eu'(\pi)(pf'(L^*)-w) = 0$$

since $Eu'(\pi)(pf'(L^*)-w)=0$ from the first-order condition for optimal labor. Therefore, it can be seen that under DARA, $-EA(\pi)u'(\pi)(pf'(L^*)-w)>0$, which is the result required.

This leads us to the following:

Proposition 1 *If the firm's utility function satisfies DARA, then optimal labor demand is decreasing in the wage rate, $L^{*\prime}(w)<0$.*

Proposition 1 is, of course, not an original result for this paper. It was shown by Batra and UIlah (1974), and probably many others have been aware of it over time. How good is DARA as a sufficient condition? That depends on how large is $-Eu'(\widetilde{p}f(L^*)-wL^*)$, since the sufficient condition given in proposition 1 is based on adding onto $-Eu'(\widetilde{p}f(L^*)-wL^*)$ something that is (under the condition of DARA) non-positive. Of course, in reality the true requirement would be that what is being added is in fact less than $Eu'(\widetilde{p}f(L^*)-wL^*)>0$. One is left wondering if there is not another sufficient condition that might, at least in some cases, be somewhat better. I now go on to analyse a second sufficient conditon for the same result to hold, that does not rely upon DARA.

3.2 Option 2

Proposition 1 shows that in order for the optimal demand for labor to be decreasing in the wage, it is sufficient that the firm's utility function satisfies DARA. Here I provide a different way of analysing the issue, that removes the requirement of DARA, but that adds a condition related to relative risk aversion.

As above, let us assume the outcome we are expecting is a negatively sloped demand for labor, and thus we can write the following equation, which is a direct consequence of (2):

$$-E\left[L^* u''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) + u'(\widetilde{p}f(L^*) - wL^*)\right] < 0$$

Multiplying by -1, this is

$$E\left[L^* u''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) + u'(\widetilde{p}f(L^*) - wL^*)\right] > 0 \tag{3}$$

We now consider what is required of the left-hand side in order for the inequality to hold true.

The easiest way forward is to see that (3) is just

$$EL^* u''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*) - w) > -Eu'(\widetilde{p}f(L^*) - wL^*)$$

Multiply the L^* term into the bracketed term on the left-hand side, so that this becomes

$$Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*)L^* - wL^*) > -Eu'(\widetilde{p}f(L^*) - wL^*)$$

Now, notice that since the production function, $f(L)$, is concave with $f(0) = 0$, it happens that $f'(L)L < f(L)$ for all positive L. That is, for all prices p we have $pf'(L^*)L^* - wL^* < pf(L^*) - wL^*$. And since $u'' < 0$, this gives us, for all p

$$u''(pf(L^*) - wL^*)(pf'(L^*)L^* - wL^*) > u''(pf(L^*) - wL^*)(pf(L^*) - wL^*)$$

Taking expectations,

$$Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f'(L^*)L^* - wL^*) > Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f(L^*) - wL^*)$$

Therefore, if $Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f(L^*) - wL^*) > -Eu'(\widetilde{p}f(L^*) - wL^*)$, then (3) is satisfied for sure. Notice that this can be written as

$$-\frac{Eu''(\widetilde{p}f(L^*) - wL^*)(\widetilde{p}f(L^*) - wL^*)}{Eu'(\widetilde{p}f(L^*) - wL^*)} < 1$$

Thus, we get:

Proposition 2 *If* $-\frac{Eu''(\widetilde{\pi})\widetilde{\pi}}{Eu'(\widetilde{\pi})} < 1$, *where* $\widetilde{\pi} = \widetilde{p}f(L^*) - wL^*$, *then the labor demand curve is decreasing in the wage rate,* $L^{*\prime}(w) < 0$.

Notice that proposition 2 does not require DARA at all. Indeed, the only way the requirement is not perfect (in the sense of being both sufficient and necessary) is that along the way there was a substitution using the concavity of the production function. Also, it is worthy to note that the function in proposition 2, $-\frac{Eu''(\widetilde{\pi})\widetilde{\pi}}{Eu'(\widetilde{\pi})}$ is suggestive of, but by no means equal to, relative risk aversion. As it happens, we can indeed show a sufficient condition that is exactly related to relative risk aversion.

3.3 Option 3

Notice that, if the term within the square brackets in (3), which is subject to the expectations operator, were to be positive for all possible values of p, then the expectation must be strictly positive. Thus, a sufficient condition for (2) is

$$L^*u''(pf(L^*) - wL^*)(pf'(L^*) - w) + u'(pf(L^*) - wL^*) > 0 \quad \forall p \tag{4}$$

Condition (4) requires a different function to be positive than what was done for proposition 1 above. Above, we required that the expectation of the first term of (4) be positive, whereas now we require that the first term be no more negative than the value of the second term for all possible outcomes of the price distribution. In one sense, (4) is more restrictive since it requires something to happen for all prices, but in another sense it is less restrictive since it does not require the term with ambiguous sign to be everywhere positive. Certainly, (4) is more restrictive than the analysis done to arrive at proposition 2. Now, divide by marginal utility, so that what we want is

$$L^* \frac{u''(pf(L^*) - wL^*)}{u'(pf(L^*) - wL^*)}(pf'(L^*) - w) > -1 \quad \forall p$$

Multiply again by -1, and simplify the notation by using A, the Arrow-Pratt measure of absolute risk aversion;

$$A(pf(L^*) - wL^*)L^*(pf'(L^*) - w) < 1 \quad \forall p$$

This is better written as

$$A(pf(L^*) - wL^*)(pf'(L^*)L^* - wL^*) < 1 \quad \forall p \tag{5}$$

Finally, recall that from the assumed concavity of the production function, $L^* < \frac{f(L^*)}{f'(L^*)}$, so

$$\begin{aligned}
A(pf(L^*) - wL^*)(pf'(L^*)L^* - wL^*) &< A(pf(L^*) - wL^*)\left(pf'(L^*)\frac{f(L^*)}{f'(L^*)} - wL^*\right) \\
&= A(pf(L^*) - wL^*)\left(pf(L^*) - wL^*\right) \\
&= R(pf(L^*) - wL^*) \qquad (6)
\end{aligned}$$

where again R denotes the Arrow-Pratt measure of relative risk aversion. Given (5) and (6), we now see that (5) will always hold if

$$R(pf(L^*) - wL^*) \leq 1 \quad \forall p$$

Thus, we have:

Proposition 3 *A sufficient condition for the optimal demand for labor to have negative slope, $L^{*\prime}(w) < 0$, is that the firm's relative risk aversion is everywhere no greater than 1.*

Again, it is interesting to notice that Proposition 3 does not require decreasing absolute risk aversion at all, only the constraint on relative risk aversion.

4 Discussion and conclusion

In this paper I have reconsidered the result from the existing literature that when a competitive firm faces a risky output price, its demand for labor will be downward sloping if the firm's utility function for money profits satisfies DARA. I have shown that alternative sufficient conditions exist that guarantee that the demand for labor is a decreasing function of the wage. One of those alternative sufficient conditions is that relative risk aversion is everywhere no greater than 1, and the other is related to that same restriction. Neither of the alternative conditions relies upon an assumption of DARA.

Katz (1983) took exception to the use of relative risk aversion in the case of firms acting in an environment of risky prices, since if profits turn out to be negative, then relative risk aversion would also be negative, which would interfere with any comparisons of relative risk aversion over different profit outcomes. However, at no point in what was done in the present paper was relative risk aversion compared over different profit levels. Indeed, if profit were to be negative for some price outcomes, then relative risk aversion would necessarily be less than 1 (since it would be negative). Thus, what is of interest here is the size of relative risk aversion when profits are positive. If the relative risk aversion function is valued no greater than 1 for any positive profit outcome, then the result of proposition 2 holds.

That relative risk aversion should be no greater than 1 (as required for proposition 3, and which would also satisfy proposition 2) might, at first blush, be considered a somewhat dubious assumption, with most estimates putting relative risk aversion greater than 1. Essentially, the condition in Proposition 3 here requires the firm to be less relative risk averse than the log function. It is also worthwhile to mention that estimates of relative risk aversion are generally restricted to individuals, not firms. The firm in the analysis here is clearly a risk-taker, and so it might be reasonable to accept that it will not be particularly risk averse.

References

[1] Baron, D. (1970), "Price Uncertainty, Utility, and Industry Equilibrium in Pure Competition", *International Economic Review*, 11; 463-80.

[2] Batra, R.N. and A. Ullah (1974), "Competitive Firm and the Theory of Input Demand under Price Uncertainty", *Journal of Political Economy*, 82(3); 537-48.

[3] Hartman, R. (1975), "Competitive Firm and the Theory of Input Demand under Price Uncertainty: Comment", *Journal of Political Economy*, 83(6); 1289-90.

[4] Hartman, R. (1976), "Factor Demand with Output Price Uncertainty", *American Economic Review*, 66(4); 675-81.

[5] Katz, E. (1983), "Relative Risk Aversion in Comparative Statics", *American Economic Review*, 73(3); 452-3.

[6] Leyland, H.E. (1972), "The Theory of the Firm Facing Uncertain Demand", *American Economic Review*, 62(3); 278-91.

[7] Sandmo, A. (1971), "On the Theory of the Competitive Firm Under Price Uncertainty", *American Economic Review*, 61(1); 65-73.