

DEPARTMENT OF ECONOMICS AND FINANCE
COLLEGE OF BUSINESS AND LAW
UNIVERSITY OF CANTERBURY
CHRISTCHURCH, NEW ZEALAND

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Microeconomics Approach**

**Richard Watt
Phillip Gunby**

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**Department of Economics and Finance
College of Business and Law
University of Canterbury
Private Bag 4800, Christchurch
New Zealand**

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Optimal Pacing of 400m and 800m Races: A Standard Microeconomics Approach

Richard Watt*¹
Phillip Gunby¹

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Abstract: The optimal way for an athlete to run 400m and 800m races has long been debated by athletes, coaches, and more recently by physicists and applied mathematicians. These two particular running events are different from both shorter and longer races, due to their emphasis upon an athlete's lactate system. If the first part of the race is run either too quickly (in which case the fatigue that is generated makes the second half of the race both slow and painful) or too slowly (in which case even though the second half will be fast, it cannot make up for the time lost earlier on),² the total time for the entire distance will not be as fast as it could otherwise have been. In this paper we discuss this choice problem in a standard microeconomic setting, and we solve for the optimal "split" times for the first and second half of the race.

Keywords: constrained optimization, microeconomics, athletics

¹ Department of Economics and Finance, University of Canterbury, Christchurch, NEW ZEALAND

²There are a great many examples of an athlete going too fast too early and losing because of extreme fatigue towards the end, but somewhat fewer of the opposite. However one very eloquent portrait of a runner going out too slowly is given by Harold Abrahams, who, after having over exerted at the start of the semi-finals of the 200m at the 1924 Olympics, only to tire excessively at the end of that race, decided to run the final with the opposite tactic. In his own words "I had a theory that I would reserve some of my speed for the last 100 yards, and I reserved so much that I finished up far down the course, with more than half of my money to spend and the bar closed." (cited from Ryan (2011), pg. 157).

* Corresponding Author: Richard Watt, Email: richard.watt@canterbury.ac.nz

Optimal pacing of 400m and 800m races

“Do you not know that in a race all the runners run, but only one gets the prize?
Run in such a way as to get the prize.” 1 Corinthians 9:24

1 Introduction

There is a now quite established literature on the mathematics of the optimal speed profile of running races (see, for example, Keller 1974, Woodside 1991, and Reardon 2013).³ The interesting cases occur at the 400m and 800m distances (respectively, one and two laps of an athletics track), with races at distances either shorter than 400m or longer than 800m being uncontroversially run at even pace. The one and two lap races are different because of the extreme lactic component that is involved in running these distances (for a very clear discussion of this point, see Prentergast 2002). The overriding conclusion of the mathematical literature is that the “optimal” profile for 400m and 800m races involves initial acceleration up to a maximal speed, which is reached early on in the race, and then constant deceleration over the remainder of the distance. The latest in this line of research is the paper by Reardon (2013), who (following the earlier work of Keller 1974 and Woodside 1991) uses dynamic optimization techniques to derive an optimal pacing strategy for both the 400m and the 800m races that returns race profiles that are extremely similar to the actual pacing strategies used in both of the current world-record runs over these two distances. In both cases, the optimal profile is characterized by continual deceleration after an initial top speed has been reached early on in the race.

There are, however, major drawbacks with the dynamic optimization methodology. First and foremost, it is not individualized, and so if it is true that different athletes (as characterized by different physiological systems) should use a different optimal race profile, any one optimal profile is not likely to be optimal for more than only a few athletes. Furthermore, no guidance is given as to how each individual athlete should determine the appropriate values of the parameters that are involved in the models, and so calculating the individual optimal

³See also the (currently) unpublished work of Whitt (2009) and Aftalion and Bonnans (2013). The use of mathematics to study athletic performances in running races was pioneered in the decade of the 1920s by the English scientist A.V. Hill (winner of the 1922 Nobel Prize in Physiology or Medicine, not to be confused with his contemporary, the English athlete A.G. Hill, who won the 800m and 1500m gold medals at the 1920 Olympics).

profile becomes largely impossible.⁴ To illustrate this point, imagine that a 17 year-old high-school athlete, who has up to that point been running long-distance events, wants to know how he should best run 400m and 800m races. Looking to the way elite athletes (e.g. world-record holders) have run in the past gives very little information, since those elite athletes are specialists in these shorter distance events, and very likely have different physiological characteristics. Specifically, the elite athletes are in those shorter events because they have a natural advantage in speed, whereas our high-school athlete's advantage is more likely to be in endurance. Second, the dynamic optimization technique is by nature very complicated, and largely unusable by coaches and athletes, who are looking for a practical answer to the question of how to distribute effort to the race. Coaches and athletes are unlikely to be interested in knowing the optimal profile function, which would dictate the speed to be run for every separate continuous point along the distance, but rather they only want to know the optimal times that should be achieved at given discrete points along the way, normally, for example, the half-way point, or perhaps the time for each full lap of the track in longer races. The dynamic optimization methodology is overly complex, and static optimization could be used to enhance the practicality of the exercise.

The optimal distribution of effort to a race distance is quite clearly a problem in constrained decision making (with the objective being the total time taken for the run, and the constraints being those imposed by the athlete's physiological characteristics), and as such, it is very natural to study it in a standard microeconomics setting. Never-the-less, the present paper is the first to approach the problem in this way. We apply standard microeconomics techniques, involving constrained optimisation, to consider the optimal distribution of pace to a given distance to be run, d , when the objective is to minimize the total time taken in the run. In contrast to the existing literature, we do not use dynamic optimization to find an optimal pacing function, but rather we concentrate on finding the optimal split times for the first and second halves of the distance to be run. We also want to attempt to provide, in as far as is possible, a methodology that has practical use for coaches and athletes. To that end, our methodology is focused on only one parameter, the value of which summarises the fundamental physiological characteristics of the athlete in as far as the lactic component of running is concerned, and which can be estimated with relative ease for actual individual athletes.

The paper is, above all, interested in the conclusion of the existing literature that the optimal way for an athlete to run the distances of 400m and 800m is to employ a profile

⁴The existing models of optimal race strategies rely upon calibration with known world-record performances to populate the model with parameter values. It is doubtful that the same parameter values would apply to lesser athletes, and therefore the optimal race profile that is found may also be largely inapplicable to non-world class athletes.

of constant deceleration after an initial top velocity has been reached. In terms of split times, the conclusion is that the optimal profile involves the first half of the race being run faster than the second half, that is, $t_1^* < t_2^*$. We find that this result is only true for a certain physiological-type, as determined by the value of the principal parameter in our model. There exist other, also commonly observed, physiological types for which the optimal profile is even pacing ($t_1^* = t_2^*$), or acceleration ($t_1^* > t_2^*$). While our model is not calibrated against top international athletes in order to find parameter values, we do present anecdotal evidence from the 2013 World Championships in Athletics to support the claim that the optimal profile depends upon physiological make-up, and that deceleration is not a generically optimal pacing strategy for minimizing the time taken to run. We also provide evidence from one well known runner that supports the critical assumption made on the physiological constraints. Throughout we are only interested in 400 and 800m running races,⁵ and so all of our comments only apply to those two, lactate-intensive, distances.⁶

2 Model

The problem is, for any given athlete, to find the optimal split times to cover the first and the second half of the distance, respectively t_1 and t_2 , such that the total time taken, $T = t_1 + t_2$, is minimized.⁷ The problem is interesting since t_2 is restricted by the value chosen for t_1 ; the smaller is t_1 (beyond some limit point), the larger will be the minimum value that can be chosen for t_2 . This rather obvious statement is, of course, due to the fatigue that is generated by running the first half of the race, which ends up compromising the athlete's ability to perform during the second half of the race. Given the complexities of that physiological restriction upon the problem, the optimal solution is certainly not trivial.⁸

⁵We did also look into swimming races to see if the theory can also be applied there. In general it seems that it may be applicable, but it happens that swimmers (a) have a significant gain on the time for the initial part of the race due to the dive, and (b) tend to have much less variance in their speeds over a race. We therefore conjecture that even in swimming races of similar duration to 400 and 800m runs, the lactate profile is significantly different.

⁶Specifically, we hypothesize how the speed of the first half of the race might compromise the ability to run fast during the second half. This compromise will exist for 400 and 800m races (and indeed, for longer distances as well), but it may not be present for shorter races.

⁷Of course, it is not true in all cases that the objective of minimizing the final time is what the athlete should do if the overriding objective is to win a race. On many occasions race tactics will dominate, and a race can be won in a slower than usual time. However, in many other situations it is true that the primary objective is to minimize total time, and so here we follow the existing literature in adopting final time as our objective variable.

⁸It is known that the overriding constraining element is the accumulation of "lactic acid" in the muscles. Lactic acid is the by-product of muscular activity, and it accumulates faster the more effort is exerted in the activity. Lactic acid accumulation inhibits the maximal use of the muscle, and so the more lactic acid that is accumulated, the less effective will be the maximum capability of the muscle. At extreme levels, the

It must be clearly noted that athletics is far from being an exact science, and as such any effort to model how a race should be run, and to make predictions about what an athlete is capable of given other performances already achieved, needs to be understood within the context of what we are trying to achieve. Of course there are a great many other factors that will both serve to determine the final outcome and the optimal way to run. Things such as the weather conditions (above all, which way the wind is blowing), how the athlete is feeling, what the other athletes do in the race, and any number of other factors, are all elements that are important but that cannot easily be incorporated into a model of optimal pacing strategies.

The problem is essentially one that fits very easily within the standard theory of micro-economic optimization. The athlete wants to run a distance d in the least amount of time possible, and in order to do that he/she has to choose the two split times, t_1 and t_2 , for the first half of the run and the second half of the run (both of distance $\frac{d}{2}$) respectively. Therefore, the athlete wants to find the vector $t = (t_1, t_2)$, such that the objective $T(t) = t_1 + t_2$ is minimized, subject to the choice lying within a feasible set, $F(t)$.

Our main task is to determine an appropriate representation of the feasible set, $F(t)$, which is the set of points (t_1, t_2) that are feasible ways of running the race. Here, by “feasible”, we are implying choices of first and second half split times that the athlete is physically able to carry out. To begin with, we can assume that there exists a known time t_m , which is a time such that it would extend the athlete’s current abilities to the absolute maximum in order to cover the distance $\frac{d}{2}$ in a single, all-out effort in the time t_m .⁹ Clearly, for the two events that we are considering, we must have $t_i \geq t_m$ for both $i = 1, 2$.¹⁰ We assume further that this is the only constraint on the first split, t_1 . The constraint on t_2 is much more complex. While it must always be true that $t_2 \geq t_m$, we also know that the second split is severely conditioned by the physiological make-up of the athlete, and the time

muscle will shut-down entirely and cease to work. To give an analogy, it is as if a man is running across a room that gradually fills with water. The increase in the water level is greater the faster the man runs, and his objective is to reach the other side of the room before the water is so deep that he can no longer run. The optimal speed to use to run across the room will depend upon the relationship between how quickly the water level rises and his initial choice of running pace. So it is with running 400m and 800m races; an overly fast first half leads to such a lactic acid build-up that the second half of the race is disappointingly slow (and painful), and the total time taken can be poor. The lactic acid problem can be mitigated by running conservatively for the first half, but then too much is left to do in the second half, and even though the second half split is fast, again the total time is poor.

⁹We may, for example, take t_m to be the athlete’s current personal best performance for a one-off run of distance $\frac{d}{2}$. However, in some cases a genuine personal best might not have been recorded, and in that case t_m would need to be (sensibly) estimated.

¹⁰The time taken for the second half of the race, t_2 , is benefitted by having a running start. This will allow for a trivially small (for the two events that we are considering in this paper) time advantage compared to t_m . This advantage is notable in 200m races, where the second half is normally run quite a bit faster than the first half, but the advantage disappears very quickly once the half-distance goes above 100m.

already taken on the first split. The faster is t_1 , the slower is the minimum feasible value for t_2 .

It seems reasonable to assume that if the athlete finds a race strategy, (t_1, t_2) that taxes him to his absolute physical limit (i.e. it is on the boundary of the feasible set), then by increasing either t_1 or t_2 , the run will be less taxing, and therefore still feasible. A sufficient condition for this to happen is that the set of feasible points $F(t)$ is strictly convex. That is, in (t_1, t_2) space, there is a convex function $t_2 = f(t_1)$ which dictates the fastest possible time for t_2 for each given t_1 . The implication of a strictly convex feasible set for the problem at hand, together with the objective of minimising total time, is that we only need be interested in the lower boundary of the set, that is, race profiles that extend the athlete's physical capabilities to the limit. Also, the assumption of strict convexity implies that the optimal race profile is unique.

In microeconomics, one very commonly used assumption for such cases is the Cobb-Douglas functional form. We assume here a particular form of the Cobb-Douglas function, known as the Stone-Geary function. Under that assumption, the lower boundary of the feasible set is given by the following relationship:

$$(t_1 - t_m)^\alpha (t_2 - t_m)^{1-\alpha} = c \quad (1)$$

where α and c are parameters that are individually specific to the athlete at any given time. It is well known that such a specification defines, in (t_1, t_2) space, a strictly negatively sloped, strictly convex function. The set of points on and above this function is our (strictly convex) feasible set $F(t)$. The parameter c is largely irrelevant in the model, as it only measures what is meant by total physical exhaustion, which will change as the athlete gets fitter. Thus, the only parameter of consequence in the model is α , which therefore summarises the lactic capabilities of the athlete.

Essentially, (1) is a production function, where the output, c , is "physical exhaustion", and there are two inputs, t_1 and t_2 . The greater is α , the greater will be the relative contribution to exhaustion of an increase in t_1 , and the less will be the relative contribution of an increase in t_2 . This is a natural definition of an athlete with a greater endurance, or ability to generate speed when (s)he is tired (i.e. in the second half of the race). Likewise, and athlete with a relatively low value of α suffers much more exhaustion in the second half than in the first for a one unit change in split time, and so would appear to be someone who is able to easily generate speed when fresh but less able to do so when tired. This would be a natural definition of an athlete with better natural speed than endurance.

Solving for t_2 , we find that the exact equation of the lower boundary of the feasible set is

$$t_2 = k (t_1 - t_m)^{-\frac{a}{1-a}} + t_m$$

where $k \equiv c^{\frac{1}{1-\alpha}}$ is a constant.

For all that follows, we define

$$\frac{\alpha}{1-\alpha} \equiv \beta$$

so that our equation for the lower boundary of the feasible set is

$$t_2 = k (t_1 - t_m)^{-\beta} + t_m \quad (2)$$

In order for this to be of any practical use, we need to find the two values β and k that are specific to the athlete in question. Since there are only two parameters to estimate, we can easily calculate them with knowledge of two specific points in the (t_1, t_2) plane through which it is assumed that the function (1) passes. Thus, we need actual data on any two performances over distance d (that is, any two observations (t_1^a, t_2^a) and (t_1^b, t_2^b) , such that $t_i^a \neq t_i^b$ for $i = 1, 2$), such that the athlete was fully extended by the end of the run (i.e. both are performances on the lower boundary of the feasible set). Together with the value t_m , the two observed performances are sufficient to locate exactly the single curve $t_2 = k (t_1 - t_m)^{-\beta} + t_m$ that passes through the observed performances.¹¹

Given two such performances, we have

$$\begin{aligned} t_2^a &= k (t_1^a - t_m)^{-\beta} + t_m \\ t_2^b &= k (t_1^b - t_m)^{-\beta} + t_m \end{aligned}$$

Solving out these two simultaneous equations, we find

$$\beta = \frac{\ln \left(\frac{t_2^b - t_m}{t_2^a - t_m} \right)}{\ln \left(\frac{t_1^a - t_m}{t_1^b - t_m} \right)} \quad (3)$$

$$k = (t_2^b - t_m) (t_1^b - t_m)^\beta \quad (4)$$

¹¹It is useful, but not absolutely necessary, that the two observed performances be sufficiently different in nature. That is, one should have a very fast first split, and a slow second split, and the other performance should be the opposite.

3 Sebastian Coe as a test

We can test the hypothesis of the suitability of a Stone-Geary restriction function by turning to observed performances of any given athlete, but one who offers a good test-bed is the legendary English athlete Sebastian Coe, and some of his performances over the 800m distance. Coe provides a great test for our hypothesis that the lower boundary of the feasible set can be expressed in Stone-Geary form, since he does have two well-known performances that are indeed very different in nature, both run when he appears to have been in similar physical condition. The first was the European Championships 800m race in 1978 (in Prague), where he ran the first half of the race in 49.3 seconds (up to then, the fastest ever known first lap of an 800m race), and then was only able to run the second half in 55.5 seconds (for a total time of 1:44.8).¹² Then, in 1980, in the famed Olympic 800m final in Moscow, he ran the first half of the race in 55.0 seconds,¹³ and the second half in 50.9 seconds, for a total time of 1:45.9. Even though to get these two performances we have to look to two different years, note that in 1980 Coe ran many 800m races, the fastest three of which were between 1:44 and 1:45, and the fastest of which was 1:44.7. Similarly in 1978, he ran the distance many times, and had 4 performances between 1:44 and 1:45. So he appears to have been in very similar condition during those two years (as opposed to 1979, where he ran an astounding world record of 1:42.4). Coe's 400m personal best time in 1978 was 47.7 seconds, and by 1980 it was 46.9 seconds (run, incidentally in 1979). He only ran two 400m races in 1978, both very early in the season when he was not at his best. In 1980 he did not run any 400m races, but in 1979 he ran four 400m races, including the English AAA Championships (where he ran his personal best time of 46.9, and when he was certainly in very good form).¹⁴ It would seem that the time of 46.9 was likely achievable in 1978, had he run the race a few more times and later on in the season when he was in better form and running at his best over 800m. Thus for this simulation we take $t_m = 46.9$.

Inputting the data from both Prague78 and Moscow80 into (2) and (3), we our two equations for β and k we find:

¹²In all of what follows, I round the electronic times, which are measured in hundredths of seconds, to the nearest tenth of a second.

¹³The Prague splits are official, but the Moscow splits have been estimated from close observation of video footage of the race.

¹⁴He also has a relay leg to his credit of 45.5 in this same period. Relay carries typically come out faster than an athlete's best race times, mainly because of the benefit of a running start.

$$\beta = \frac{\ln\left(\frac{50.9-46.9}{55.5-46.9}\right)}{\ln\left(\frac{49.3-46.9}{55.0-46.9}\right)} = 0.62929$$

$$k = (50.9 - 46.9)(55 - 46.9)^{0.62929} = 14.920$$

Therefore, our estimate for Coe's constraining equation is

$$t_2 = 14.920 (t_1 - 46.9)^{-0.62929} + 46.9$$

Figure 1 shows a graph of this exact equation, together with the two points used to define the curve. The curve is asymptotic to the value $t_m = 46.9$ on both axes. The feasible set for Coe, based upon these two performances, is the zone of points on and above the curve.

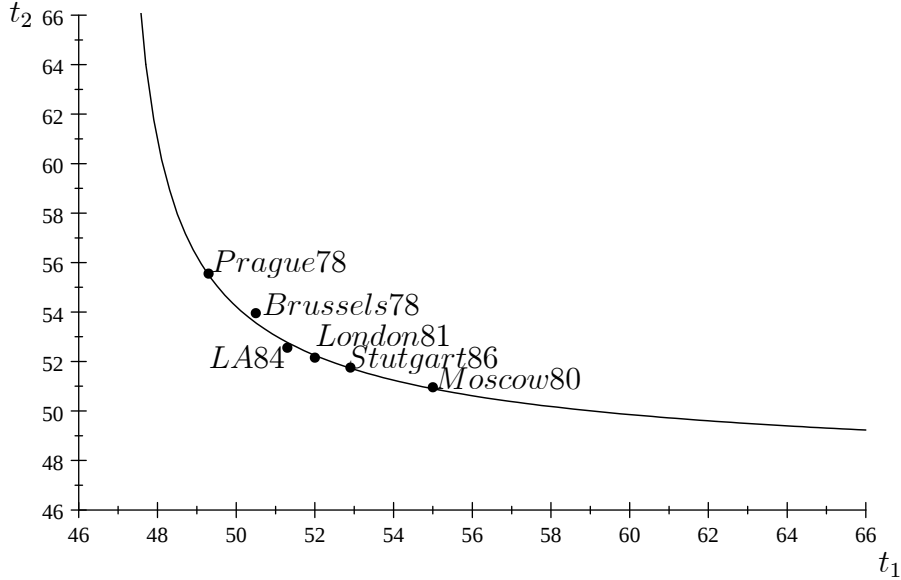


Figure 1: Estimated feasible set for Sebastian Coe

Figure 1 also shows four further points, labeled Brussels78, Stuttgart86, London81 and LA84, corresponding to other races that Coe ran and for which split data is available.¹⁵ Of particular interest are Brussels78 and London81, both of which were run in the same years

¹⁵There are two other races for which splits are available - the two 800m world records run by Coe. The first of these was run in 1979 (total time 1:42.4), and the second was run late in the season of 1981 (total time 1:41.7). Both of these are clear outliers when compared to the rest of Coe's many races, and so we do not take them into account for the analysis. Aside from those two particular efforts, he never again managed to run below 1:43 in a huge number of attempts over his career.

as the points used to calculate the frontier curve. In particular, Brussels78 was run within 2 weeks of Prague78, and un Brussels Coe ran the first 400m in 50.5, and the second in 53.8 for a total time of 1:44.3. The prediction from the curve for a race run with a first lap in 50.5 is

$$14.920 (50.5 - 46.9)^{-0.62929} + 46.9 = 53.563$$

which predicts a time of 1:44.1 (only 2 tenths of a second from what he actually achieved). The degree to which the curve predicts several other of Coe's races during that year gives credence to the plausibility of our Stone-Geary hypothesis for modelling the restriction on the second-half split.

4 Optimal splits

Go back to equation (2). This defines a curve that is the lower boundary of a strictly convex set of points that we are associating as being the feasible set for the choice problem. The objective function, as represented in the space (t_1, t_2) is a straight line with slope equal to -1 . A faster total time T is associated with a line that is closer to the origin of the graph. It is therefore obvious that the optimal solution is the point on the boundary of the feasible set that has slope equal to -1 .¹⁶

The first derivative of (1) with respect to t_1 is

$$\frac{dt_2}{dt_1} = -\beta k (t_1 - t_m)^{-\beta-1} = -\beta k (t_1 - t_m)^{-(\beta+1)}$$

Thus, the optimal value¹⁷ of t_1 , which we denote by t_1^* is the solution to

$$-\beta k (t_1^* - t_m)^{-(\beta+1)} = -1$$

It is straight-forward to solve this equation to get

$$t_1^* = \left(\frac{1}{\beta k} \right)^{-\frac{1}{\beta+1}} + t_m \quad (5)$$

From (5) we can derive the following:

Proposition 1 *If $\beta = 1$ then the optimal profile involves even splits, that is $t_1^* = t_2^*$. If $\beta < 1$ then the optimal profile involves positive splits, that is $t_1^* < t_2^*$. If $\beta > 1$ then the*

¹⁶Of course, we could also easily carry out the implied constrained optimization using the standard Kuhn-Tucker methodology, but the solution is so obvious that there is little point in doing this.

¹⁷It is easy to check that the second-order condition holds.

optimal profile involves negative splits, that is $t_1^* > t_2^*$.

Proof. From (2) and (5), the optimal second half split is

$$t_2^* = k(t_1^* - t_m)^{-\beta} + t_m = k \left(\left(\frac{1}{\beta k} \right)^{-\frac{1}{\beta+1}} \right)^{-\beta} + t_m$$

This simplifies to

$$t_2^* = k \left(\frac{1}{\beta k} \right)^{\frac{\beta}{\beta+1}} + t_m$$

Thus, we get $t_1^* \gtrless t_2^*$ as

$$\left(\frac{1}{\beta k} \right)^{-\frac{1}{\beta+1}} + t_m \gtrless k \left(\frac{1}{\beta k} \right)^{\frac{\beta}{\beta+1}} + t_m$$

which (again, after a few straight-forward simplifications) reduces to

$$\beta \gtrless 1$$

Since $\beta = \frac{\alpha}{1-\alpha}$, the condition $\beta \gtrless 1$ is equivalent to $\alpha \gtrless \frac{1}{2}$. ■

As we have noted above, the value of the constant c plays no part in the optimal choice of splits. All that matters is α .

5 Coe's optimal split

We can begin by noting that we have calculated for Coe that $\beta = 0.62929 < 1$, so he should optimally be a positive splitter. The optimal split is given by equation (5) as

$$t_1^* = \left(\frac{1}{\beta k} \right)^{-\frac{1}{\beta+1}} + t_m$$

Substituting in the values for β and k already calculated for Coe, we get¹⁸

$$t_1^* = \left(\frac{1}{0.62929 \times 14.92} \right)^{-\frac{1}{1.62929}} + 46.9 = 50.853$$

With an optimally run first 400m, equation (2) then gives us his optimal second 400m as

¹⁸Interestingly, in his first biography (Coe and Millar, 1981), Coe discusses his 1979 world record over 800m, in which the first lap was passed in 50.6 seconds, with the following words; “I just followed the pace for the first lap which after Prague, as it turned out, was exactly what Peter and I had calculated to optimum would be.” In this quote, “Peter” refers to Coe’s father and coach. No further information is given as to exactly how this “optimum” had been calculated.

$$\begin{aligned}
t_2 &= k(t_1 - t_m)^{-\beta} + t_m \\
&= 14.92 \times (50.853 - 46.9)^{-0.62929} + 46.9 \\
&= 53.182
\end{aligned}$$

which gives a total time of 1:44.0.

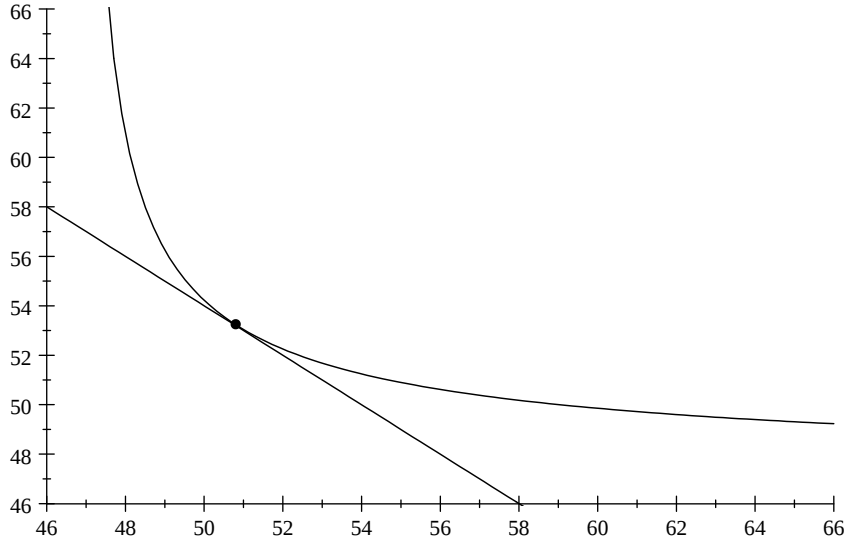


Figure 2: Sebastian Coe's optimal solution

Interestingly, in 1978 Coe did run 1:43.95 (which would be rounded to 1:44.0), but unfortunately we have not been able to find any evidence at all on the lap split that was used in that race.

6 Other optimal solutions

Proposition 1 tells us that there might be some athletes for whom negative (or even splitting) is optimal. Those are the athletes with values of β greater than or equal to 1. As discussed above, we hypothesise that β increases with the endurance base of an athlete, so that athletes who are strongly endurance-based, as opposed to strongly speed-based, are more likely to have values of β greater than 1. This would lead to those athletes optimally running the second half of a 400 or 800 meter run faster than the first half. We can gather some supporting evidence for this claim by looking at how different athletes (differentiated by their endurance

base) choose to run a single 400m. The hypothesis is that as the endurance base of the athletes in question rises (at the expense of their speed base), we should see a larger fraction of them deciding to negative split. For example, since the endurance base required to compete as a specialist increases with the distance to be run, we should see a greater fraction of negative split runs in 800m races than in 400m races.¹⁹ Indeed, Reardon (2013) has looked at the frequency of negative splits in world record runs for which official half-way split times were reliably recorded. Aggregating men and women, for the 400m there have been 34 such records, all of which were positive split. On the other hand, for the 800m there have been 42 such records, of which 39 (92.8%) were positive split.²⁰

As a further test of this hypothesis, we consider the last 400m of all of the races at the 2013 world champs of non-obstacle²¹ races of 1 lap or longer. The underlying assumption is that once the race gets into the last lap, any tactical concerns have all played out, and all athletes have the same strategy, which is to run the last lap as fast as possible. Of course, different athletes will start that last lap in different states of lactic acid accumulation (depending on their physiological make-up, their training, their mental state, and any number of other things), but that does not alter the idea that given their speed reserves at the start of the last lap, it is in their best interests to distribute those reserves over the lap such that the lap is run in the fastest possible time. If our hypothesis is correct, then we should see the frequency of negative splitting over the two 200m sections of the last lap increasing as the length of the overall race increases (i.e. as the athletes involved become more and more endurance based). The data (which is calculated from statistics available from the specialist magazine *Track and Field News*), is given in Table 1 and Figure 3.

¹⁹This general theory would also explain why it is relatively uncontroversial that distances longer than 800m, which are heavily dominated by endurance based athletes, should be optimally run with even, or perhaps negative, splits.

²⁰Of the 40 mile records, 60% were positive split, while of the 42 records for 10000m, only 52.4% were positive split.

²¹The two obstacle races that we do not consider are the 400m hurdles, and the 3000m steeplechase.

Table 1: 200m split data for the last laps of finals at 2013 World Athletics Championships²²

	ave 1st 200	ave 2nd 200	ave differential	n ^o non-positive splitters
400 M	21.6625	22.9625	1.3	0/8
400 W	23.6375	26.625	2.9875	0/8
800 M	26.6	27.225	0.625	4/8
800 W	30.3875	30.825	0.4375	4/8
1500 M	27.2125	26.475	-0.7375	8/8
1500 W	30.3833	29.9833	-0.4	5/6
5000 M	26.8666	27.2666	0.4	3/6
5000 W	32.1333	31.9166	-0.2166	5/6
10000 M	28.075	27.2	-0.875	3/4
10000 W	30.925	30.9	-0.025	2/4

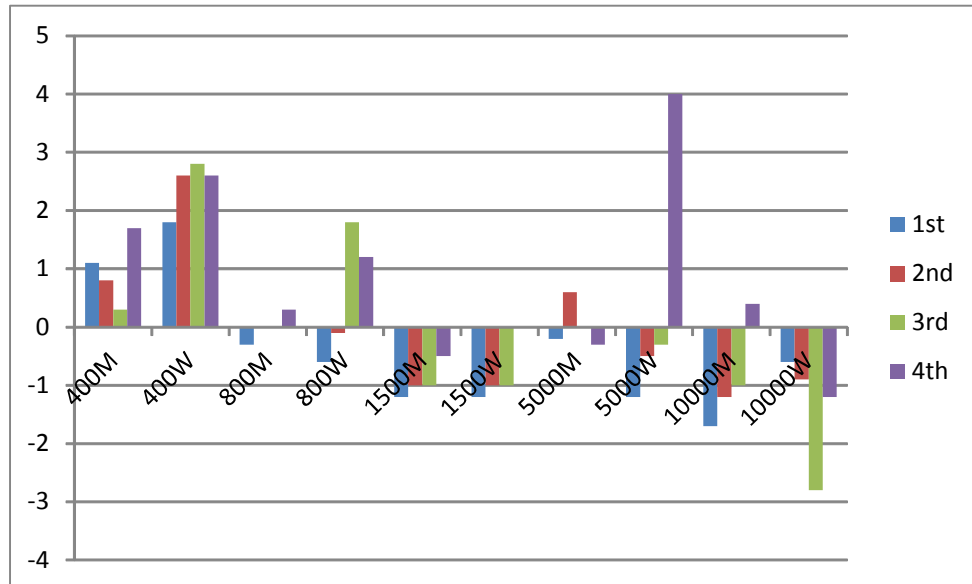


Figure 3: Second 200m time less first 200m time of last lap of races of at least 400m at 2013 World Athletics Championships (top 4 finishers)

Table 1 is fairly conclusive. All of the runners in the 400m finals (men and women) utilized a strategy of positive splitting, as one might expect of the more speed based athletes that populate that event. But as the distance of the event increases, so in general does the

²²The data source gives information on the first 8 runners for 400m, 800m for both men (M) and women (W), the first 8 runners in 1500m men, the first 6 in 1500m W, 5000m M and W, and the first 4 runners for 10000m M and W.

proportion of non-positive splitting athletes. Thus, this evidence points to more endurance based athletes tending to find that negative (or even) splitting is an optimal strategy.

Aside from the 4th place-getter in the women’s 5000m (who clearly ran a poorly judged race - see Figure 3), the overriding profile for races at 1500m and above is to negative split (second 200m time less first 200m time is negative). The unanimous profile for 400m runners is to positive split, and the 800m runners are more evenly bunched – some are positive splitters, others even splitters, and still others are negative splitters. Of course, it needs to be reiterated clearly that this is only circumstantial evidence, and by no means is this data presented as a claim for any sort of general result.

7 Conclusions

The existing literature on the optimal pace strategy to run 400m and 800m distances concludes that an athlete with the objective of running the distance in the least possible time should use a strategy of positive splitting (i.e. running the first half of the race faster than the second). In this paper, using a standard microeconomics model, we find that the choice between positive splitting, negative splitting, or even splitting depends critically upon the physiological make-up of the athletes. More speed-based athletes should probably use positive splits, but heavily endurance-based athletes might find that their optimal strategy is not one of positive splitting. Thus, in as much as most runners who participate in 400m races are speed based, it is true that most 400m runners should probably use a strategy of positive splitting. But this result does not carry over to *any* athlete wanting to run the 400m race as fast as possible. An analogous comment applies to the 800m run as well.

Furthermore, the paper provides a simple and practical test, that can be used to individualize the maximization problem to the characteristics of any given athlete. This contrasts with the existing literature, which is based squarely upon finding parameter values from calibration against only top international runners. Here, we have *exemplified* the methodology with data from a top international runner (Sebastian Coe), but the methodology itself is not specific to that athlete. Rather, any athlete can find their own parameter values, fit them to the model, and output their own individual optimal split for the half-way point in a race.

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