DEPARTMENT OF ECONOMICS AND FINANCE

COLLEGE OF BUSINESS AND ECONOMICS

UNIVERSITY OF CANTERBURY

CHRISTCHURCH, NEW ZEALAND

Allocative Downside Risk Aversion

Richard Watt and Francisco J. Vazquez

WORKING PAPER

No. 61/2010

Department of Economics and Finance
College of Business and Economics
University of Canterbury
Private Bag 4800, Christchurch
New Zealand

Allocative Downside Risk Aversion

Richard Watt (University of Canterbury)*
Francisco J. Vazquez (Universidad Autonoma de Madrid)

May 14, 2010

Abstract

Traditionally, downside risk aversion is the study of the placement of a pure risk (a secondary risk) on either the upside or the downside of a primary two-state risk. When the decision maker prefers to have the secondary risk placed on the upside rather than the downside of the primary lottery, he is said to display downside risk aversion. The literature on the intensity of downside risk aversion has been clear on the point that greater prudence is not equivalent to greater downside risk aversion, although the two concepts are linked. In the present paper we present a new, and we argue equally natural, concept of the downside risk aversion of a decision maker, namely the fraction of a zero mean risk that the decision maker would optimally place on the upside. We then consider how this measure can be used to identify the intensity of downside risk aversion. Specifically, we show that greater downside risk aversion in our model can be accurately measured by a relationship that is very similar to, although somewhat stronger than, greater prudence.

JEL: D8

Keywords: risk aversion, prudence, downside risk

*Corresponding author. richard.watt@canterbury.ac.nz; tel. (+64) 0210403497

Allocative Downside Risk Aversion

1 Introducion

Downside risk aversion (DSRA) has now become a standard inclusion in the literature of the study of choice under risk. It is, therefore, natural that a measure of the intensity of downside risk aversion be known. This has proved to be a rather more difficult task than would be expected, and more than one measure has been suggested. Concretely, two measures of local downside risk aversion can be found in the literature - the ratio of third to first derivates of utility (see for example, Modica and Scarsini 2005, and Crainich and Eeckhoudt 2007), and the "Schwartzian derivative" (see, for example, Keenan and Snow 2002, 2009a and 2009b).[1] Globally, Keenan and Snow (2009b) show that the intensity of downside risk aversion is found by a positive third derivative of the function that transforms one utility function (the less downside risk averse one) into another (the more downside risk averse one).

In the present paper we reconsider the issue of a measure of downside risk aversion, and we provide a new concept of how we can understand what downside risk aversion is. We argue that this new definition is equally natural as that which is currently in use. Then we consider some comparative statics of this new definition of downside risk aversion, and above all we consider how utility functions can be ranked according to the new definition of intensity of downside risk aversion. Concretely, we find that intensity can be reliably measured by a concept that is very similar to prudence.

2 Traditional downside risk aversion

Traditionally, the literature on DSRA starts with a comparison of two lotteries, each of which contains two independent risks. In both lotteries, the decision maker faces a first risk which is a one-half chance of the loss of x_2 and a one-half chance of a gain of x_1, where $x_i \geq 0$ $i = 1, 2$, and $x_i > 0$ for at least one $i = 1, 2$.[2] We denote this as the "x risk", or the "primary risk". Then, a second lottery, defined by a random variable $\widetilde{y}$, which is characterized by having zero mean $E\widetilde{y} = 0$ and postive variance (we denote this second risk as the "$\widetilde{y}$ risk", or the "secondary risk"), is placed either on the "upside" or the "downside" of the first risk. That is, the decision maker is asked to rank the two following expected utility measures[3]:

[1] Although Menezes et al. (1980) identifyed prudence as aversion to downside risk, prudence cannot be shown to measure the intensity of downside risk aversion. In fact, since the ratio of third to first derivatives of utility is the product of prudence and risk aversion, the Crainich and Eeckhoudt measure uses, implicitly, prudence.

[2] In several studies, x_1 is set at 0. Really, this restriction is not needed - all that is required is that the lottery has an upside and a downside.

[3] Here, u represents a strictly increasing and concave Von Neumann-Morgenstern indirect utility function for wealth, and w is an amount of non-random initial wealth, which is assumed

1. $\widetilde{y}$ is on the upside, giving expected utility of $\frac{1}{2}Eu(w+x_1+\widetilde{y})+\frac{1}{2}u(w-x_2)$.
2. $\widetilde{y}$ is on the downside, giving expected utility of $\frac{1}{2}u(w+x_1)+\frac{1}{2}Eu(w-x_2+\widetilde{y})$.

It is not difficult to show that (see Appendix 1), conditional upon marginal utility being convex (i.e. $u'''>0$), then the expected utility of having $\widetilde{y}$ on the upside is always greater than the expected utility of having it on the downside. That is

$$u'''>0 \Rightarrow \frac{1}{2}Eu(w+x_1+\widetilde{y})+\frac{1}{2}u(w-x_2) > \frac{1}{2}u(w+x_1)+\frac{1}{2}Eu(w-x_2+\widetilde{y}). \quad (1)$$

A decision maker displaying such preferences is then characterized as suffering downside risk aversion.

In order to measure the intensity of downside risk aversion in terms of the shape of the utility function u, the traditional strategy has been to define a concept that is akin to a risk premium, such that the inequality in (1) is replaced by an equality. This, for example is the methodology adopted by Crainich and Eeckhoudt (2007). Imagine that the secondary risk was intially placed on the downside, and that the decision maker is compensated by a sure-thing payment of, say, m, in the good state such that the decision maker is indifferent to the lottery in which the zero mean risk is placed on the upside. That is, m must satisfy

$$\frac{1}{2}Eu(w+x_1+\widetilde{y})+\frac{1}{2}u(w-x_2)=\frac{1}{2}u(w+x_1+m)+\frac{1}{2}Eu(w-x_2+\widetilde{y}). \quad (2)$$

Using a second-order Taylor expansion, Crainich and Eeckhoudt (2007) show that m can be expressed as a function of $\frac{u'''}{u'}$, and therefore $\frac{u'''}{u'}$ can be taken as being a measure of the intensity of downside risk aversion, since the greater is $\frac{u'''}{u'}$, the greater would have to be the upside compensation, m, for having the risk located on the downside.[4]

3 A new downside risk aversion methodology: allocative downside risk aversion

Imagine that, instead of looking for a simple preference of where to locate the secondary lottery as above, we ask the following question of our decision maker. Given the choice, what fraction of the zero mean risk would you like to locate on the upside of the primary risk? That is, we allow our decision maker to locate a

to be greater than or equal to 0.

[4]There is, actually, a dificulty with the Crainich and Eeckhoudt approach. Concretely, although the left-hand-side of (1) is greater than its right-hand-side, and introducing the compensation m we can increase the value of the right-hand-side, it is not clear that the the equality that we search for can ever be achieved. That is, there is no guarantee that there actually exists an m that satisfies (2). In fact, it can be shown that it is generally true that there is no universal solution. This issue, however, will not be addressed in the current paper.

fraction λ of $\widetilde{y}$ on the upside, and thus a fraction $(1-\lambda)$ on the downside, and then ask what is the optimal value of λ?

To study this choice, we define the indirect utility function

$$\begin{aligned} U(\lambda) &= \frac{1}{2}Eu(w_1+\lambda\widetilde{y}) + \frac{1}{2}Eu(w_2+(1-\lambda)\widetilde{y}) \\ &= \frac{1}{2}\left[Eu(w_1+\lambda\widetilde{y}) + Eu(w_2+(1-\lambda)\widetilde{y})\right] \end{aligned}$$

where $w_1 = w + x_1$ and $w_2 = w - x_2$, so that $w_1 > w_2$.

The first-order condition for an optimal choice of λ is

$$U'(\lambda^*) = 0 \Longrightarrow \frac{1}{2}\left[Eu'(w_1+\lambda^*\widetilde{y})\widetilde{y} - Eu'(w_2+(1-\lambda^*)\widetilde{y})\widetilde{y}\right] = 0$$

The second order condition, $U''(\lambda^*) < 0$ is satisfied by concavity of u.

Note that the first-order condition can be expressed as

$$\lambda^* \leftarrow Eu'(w_1+\lambda^*\widetilde{y})\widetilde{y} - Eu'(w_2+(1-\lambda^*)\widetilde{y})\widetilde{y} = 0$$

Define $G(\lambda) \equiv Eu'(w_1+\lambda^*\widetilde{y})\widetilde{y} - Eu'(w_2+(1-\lambda^*)\widetilde{y})\widetilde{y}$. By definition, we now have $G(\lambda^*) = 0$. Consider $G(0)$;

$$\begin{aligned} G(0) &= Eu'(w_1)\widetilde{y} - Eu'(w_2+\widetilde{y})\widetilde{y} \\ &= u'(w_1)E\widetilde{y} - Eu'(w_2+\widetilde{y})\widetilde{y} \\ &= -Eu'(w_2+\widetilde{y})\widetilde{y} \end{aligned}$$

where we have used the fact that $E\widetilde{y} = 0$.

Now, from elementary statistics, we know that, for any two random variables $\widetilde{a}$ and $\widetilde{b}$ it is true that $cov(\widetilde{a},\widetilde{b}) = E\widetilde{a}\widetilde{b} - E\widetilde{a}E\widetilde{b}$. That is, $E\widetilde{a}\widetilde{b} = cov(\widetilde{a},\widetilde{b}) + E\widetilde{a}E\widetilde{b}$. Given that, we know that

$$\begin{aligned} Eu'(w_2+\widetilde{y})\widetilde{y} &= cov(u'(w_2+\widetilde{y}),\widetilde{y}) + Eu'(w_2+\widetilde{y})E\widetilde{y} \\ &= cov(u'(w_2+\widetilde{y}),\widetilde{y}) \end{aligned}$$

where again we have used the fact that $E\widetilde{y} = 0$.

Under concavity of u, it must be true that the greater is y, the smaller is $u'(w_2+y)$, that is, $cov(u'(w_2+\widetilde{y}),\widetilde{y}) < 0$. Finally, then, we note that $G(0) = -Eu'(w_2+\widetilde{y})\widetilde{y} = -cov(u'(w_2+\widetilde{y}),\widetilde{y}) > 0$. So the slope of the first derivative of the indirect utility function at $\lambda = 0$ is strictly positive.

Using an identical analysis, it is straight forward to show that $G(1) < 0$, that is, the slope of the first derivative of the indirect utility function at $\lambda = 1$ is strictly negative. Together with the strict concavity of the indirect utility function, we now know that the following is true:

Lemma 1 $0 < \lambda^* < 1$, *that is, neither extreme is a solution to the problem.*

Note that this result is due only to concavity of the utility function, not to convexity of marginal utility.

In and of itself, lemma 1 is interesting. It says that moving all of the zero-mean risk to the upside of the primary risk is not actually an optimal allocation for the decision maker. Or in other words, a comparison between having the zero mean risk on either the upside or the downside, as is traditionally done in the downside risk aversion literature, is a comparison of two sub-optimal risk allocations.

In what follows, the following lemma will be useful on several occasions:

Lemma 2 *Let $h(w+\lambda y)$ be an increasing (resp. decreasing) function, with $w > 0$ and $\lambda > 0$, and let $\widetilde{y}$ be a zero mean random variable. Then $Eh(w+\lambda\widetilde{y})\widetilde{y} > 0$ (resp. < 0).*

Proof. We prove here the case of h increasing. For any $y < 0$ it is true that $h(w + \lambda y) < h(w)$, and for any $y > 0$ it is true that $h(w + \lambda y) > h(w)$. But then,

$$\begin{aligned}\forall y &< 0 \;\; h(w + \lambda y)y > h(w)y \\ \forall y &> 0 \;\; h(w + \lambda y)y > h(w)y\end{aligned}$$

Thus, we have $\forall y \;\; h(w + \lambda y)y \geq h(w)y$, with equalty only when $y = 0$. Given that, take expectations over $\widetilde{y}$, to get $Eh(w + \lambda\widetilde{y})\widetilde{y} > h(w)E\widetilde{y} = 0$. The case of h decreasing is proved in an analogous manner. ■

We now state and prove the following theorem:

Theorem 1 *Assuming that the decision maker suffers from downside risk aversion (that is, marginal utility is convex), the optimal risk allocation is characterized by $\lambda^* > \frac{1}{2}$, that is, more than half of the zero mean risk is held on the upside.*

Proof. We only need to prove that $G\left(\frac{1}{2}\right) > 0$. That is, we need to show that $Eu'(w_1 + \frac{1}{2}\widetilde{y})\widetilde{y} - Eu'(w_2 + \frac{1}{2}\widetilde{y})\widetilde{y} > 0$, i.e. that $Eu'(w_1 + \frac{1}{2}\widetilde{y})\widetilde{y} > Eu'(w_2 + \frac{1}{2}\widetilde{y})\widetilde{y}$. Consider how the function $Eu'(w + \frac{1}{2}\widetilde{y})\widetilde{y}$ changes with w. The derivative with respect to w is $Eu''(w + \frac{1}{2}\widetilde{y})\widetilde{y}$. Since we are assuming that marginal utility is convex, we must have $u''' > 0$, that is, u'' is an increasing function. Thus, from Lemma 2, $Eu''(w+\frac{1}{2}\widetilde{y})\widetilde{y} > 0$, or in short, $Eu'(w+\frac{1}{2}\widetilde{y})\widetilde{y}$ is an increasing function of w. Thus $Eu'(w_1 + \frac{1}{2}\widetilde{y})\widetilde{y} > Eu'(w_2 + \frac{1}{2}\widetilde{y})\widetilde{y}$ which was what was required to be shown. ■

This theorem provides us with a new definition of downside risk aversion; if the optimal choice of risk allocation is to place more than half of the zero mean risk on the upside of the other risk, then the decision maker displays downside risk aversion. In order to differentiate with the traditional measure of downside risk aversion, and to highlight the fact that our measure of downside risk aversion is founded upon an optimal risk allocation, we define "allocative" downside risk aversion:

Definition 1 *If $\lambda^* > \frac{1}{2}$, the decision maker displays allocative downside risk aversion.*

It also seems natural to say that the greater is λ^*, i.e. the more of the zero mean risk the decision maker would like to place on the upside, the greater is the intensity of allocative downside risk aversion that this decision maker displays. Indeed, we will also define

Definition 2 *If $\lambda_a^* > \lambda_b^* > \frac{1}{2}$, then decision maker "a" displays greater allocative downside risk aversion than does individual "b".*

4 Comparative statics

In this section we will discuss some comparative statics issues. Firstly, we will show how λ^* changes with the parameters of the other risk, that is, x_1 and x_2. Secondly, we will consider the relationship between λ^* and the characteristics of the utility function. Throughout we retain the assumption that marginal utility is convex (i.e. $u''' > 0$) in order that the decision maker is indeed downside risk averse.

Theorem 2 *An increase in either x_1 or x_2 leads to a greater optimal value of λ.*

Proof. From the implicit function theorem, and the fact that the second derivative of the objective function is negative from the second order condition, the sign of $\frac{\partial \lambda^*}{\partial x_1}$ is the same as the sign of $\frac{\partial G}{\partial x_1}$, where $G(\lambda) \equiv Eu'(w + x_1 + \lambda^* \widetilde{y})\widetilde{y} - Eu'(w - x_2 + (1 - \lambda^*)\widetilde{y})\widetilde{y}$. Carrying out the implied derivative yields

$$\frac{\partial G}{\partial x_1} = Eu''(w + x_1 + \lambda^* \widetilde{y})\widetilde{y}$$

However, under the condition that marginal utility is convex, u'' is an increasing function, and so from Lemma 2 $Eu''(w+x_1+\lambda^*\widetilde{y})\widetilde{y} > 0$. Thus, we have $\frac{\partial \lambda^*}{\partial x_1} > 0$. The proof of $\frac{\partial \lambda^*}{\partial x_2} > 0$ is done in an analogous manner. ■

This theorem points to an interesting aspect of downside risk aversion. When the primary risk (the x risk) becomes more risky,[5] then the optimal allocative response is to place a greater part of the secondary risk (that defined by $\widetilde{y}$) on the upside. That is, a riskier x lottery leads to greater allocative downside risk aversion. This result can be related to the concept of temperance under a background risk (see Gollier and Pratt, 1996), and it shows that allocative downside risk aversion is aggrevated by an increase in the primary risk.

Secondly, what happens to the optimal risk allocation as the size of risk-free wealth, w, increases? As it happens, we can sign this effect only under a dubious assumption on the higher order derivatives of utility. Concretely, we have the following result:

[5] The x risk becomes more risky when $x_1 - x_2$ increases, which occurs if either x_1 or x_2 increases. Since the probability of each x_i is set at one-half, it is not possible to alter the riskiness of the x lottery by changing its probabilities.

Theorem 3 *If $u''''(z) > 0$, then λ^* increases with w.*

Proof. From the implicit function theorem, and the first-order condition for λ^*, and recalling that $w_1 = w + x_1 > w_2 = w - x_2$, we have

$$\text{sign}\frac{\partial \lambda^*}{\partial w} = \text{sign}\left[Eu''(w_1 + \lambda^* \widetilde{y})\widetilde{y} - Eu''(w_2 + (1-\lambda^*)\widetilde{y})\widetilde{y}\right]$$

Define $G(\lambda) \equiv Eu''(w_1 + \lambda\widetilde{y})\widetilde{y} - Eu''(w_2 + (1-\lambda)\widetilde{y})\widetilde{y}$, so that the expression on the right-hand-side is just $G(\lambda^*)$. At $\lambda = 1$, we get $G(1) = Eu''(w_1 + \widetilde{y})\widetilde{y} > 0$, where the sign is given by lemma 2 and the fact that u'' is an increasing function. Secondly, at $\lambda = \frac{1}{2}$, we get $G(\frac{1}{2}) = Eu''(w_1 + \frac{1}{2}\widetilde{y})\widetilde{y} - Eu''(w_2 + \frac{1}{2}\widetilde{y})\widetilde{y}$. However, consider $H(z) = Eu''(z + \frac{1}{2}\widetilde{y})\widetilde{y}$. We have $H'(z) = Eu'''(z + \frac{1}{2}\widetilde{y})\widetilde{y}$. In order to sign this, we need to know if u''' is increasing or decreasing, and then we can use Lemma 2. Concretely, if u''' is increasing (i.e. $u'''' > 0$), then $H'(z) = Eu'''(z + \frac{1}{2}\widetilde{y})\widetilde{y} > 0$, and $G(z)$ would be an increasing function of z. In this case (since $w_1 > w_2$), we would have $Eu''(w_1 + \frac{1}{2}\widetilde{y})\widetilde{y} - Eu''(w_2 + \frac{1}{2}\widetilde{y})\widetilde{y} > 0$. Given that, in the case of $u'''' > 0$, it would be true that $G(\lambda) > 0$ for all λ between one-half and one, if $G(\lambda)$ were a monotone function. But since $G'(\lambda) = Eu'''(w_1 + \lambda\widetilde{y})\widetilde{y}^2 + Eu'''(w_2 + (1-\lambda)\widetilde{y})\widetilde{y}^2 > 0$, it turns out that G is indeed monotone (increasing, in fact), and so in this case we would have $G(\lambda^*) > 0$, and correspondingly $\frac{\partial\lambda^*}{\partial w} > 0$. ■

A similar result applies if we consider the effect of an increase in the size of the risk $\widetilde{y}$. To do so, set $\widetilde{y} = k\widetilde{s}$, with $k > 0$. Then an increase in k corresponds to an increase in the size of the risk $\widetilde{y}$.

Theorem 4 *If $u'''' > 0$, then λ^* increases with k.*

Proof. From the implicit function theorem, and the first-order condition for λ^*, we have

$$\text{sign}\frac{\partial \lambda^*}{\partial k} = \text{sign}\left[Eu''(w_1 + \lambda^* k\widetilde{s})\lambda^* k\widetilde{s} - Eu''(w_2 + (1-\lambda^*)k\widetilde{s})(1-\lambda^*)k\widetilde{s}\right]$$

Now define $G(\lambda) \equiv Eu''(w_1 + \lambda k\widetilde{s})\lambda k\widetilde{s} - Eu''(w_2 + (1-\lambda)k\widetilde{s})(1-\lambda)k\widetilde{s}$, so that the expression on the right-hand-side is just $G(\lambda^*)$. At $\lambda = 1$, we get $G(1) = Eu''(w_1 + k\widetilde{s})k\widetilde{s} > 0$, where the sign is given by lemma 2 and the fact that u'' is an increasing function. Secondly, at $\lambda = \frac{1}{2}$, we get $G(\frac{1}{2}) = Eu''(w_1 + \frac{1}{2}k\widetilde{s})\frac{1}{2}k\widetilde{s} - Eu''(w_2 + \frac{1}{2}k\widetilde{s})\frac{1}{2}k\widetilde{s}$. Now consider $H(z) = Eu''(z + \frac{1}{2}k\widetilde{s})\frac{1}{2}k\widetilde{s}$. We have $H'(z) = Eu'''(z + \frac{1}{2}k\widetilde{s})\frac{1}{2}k\widetilde{s}$. If u''' is increasing (i.e. $u'''' > 0$), then $H'(z) > 0$, and $G(z)$ would be an increasing function of z. In this case (since $w_1 > w_2$), we would have $G(\frac{1}{2}) > 0$. Given that, in the case of $u'''' > 0$, it would be true that $G(\lambda) > 0$ for all λ between one-half and one, if $G(\lambda)$ were a monotone function. But since $G'(\lambda) = Eu'''(w_1 + \lambda k\widetilde{s})\lambda(k\widetilde{s})^2 + Eu'''(w_2 + (1-\lambda)\widetilde{y})(1-\lambda)(k\widetilde{s})^2 + Eu''(w_1 + \lambda k\widetilde{s})k\widetilde{s} + Eu''(w_2 + (1-\lambda)k\widetilde{s})k\widetilde{s} > 0$ (where again we have used the fact that u'' is an increasing function and lemma 2) it turns out that G is indeed monotone (increasing, in fact), and so in this case we would have $G(\lambda^*) > 0$, and correspondingly $\frac{\partial\lambda^*}{\partial k} > 0$. ■

The assumption needed to derive a crisp comparative static result for risk-free wealth and the size of the allocatable risk, that the fourth derivative of utility is positive, can be thought to be somewhat dubious. It is more common to see the assumption that the derivatives of utility alternate in sign, with the negatively numbered derivatives (i.e. first, third, etc.) being positive in sign, and the positively numbered derivatives (i.e. second, fourth, etc.) being negative in sign. Concretely, a positive fourth derivative implies that absolute prudence is not necessarily decreasing in wealth, but does not alter the more common assumptions of positive and decreasing absolute risk aversion.

Indeed, it is well known that in order to get an unambiguous wealth effect in ordinarly problems in which a single risk is to be undertaken, one needs to make a non-standard assumption on the third derivative of utility - concretely that it is negative, which would in turn imply that the decision maker would not suffer downside risk aversion. A positive fourth derivative, while certainly not the most comfortable of assumptions, is certainly far less obtrusive than non-standard assumptions on the third derivative.

Finally, we now consider the relationship between allocative downside risk aversion and the shape of the utility function u, in order to discuss the intensity of allocative downside risk aversion. This is of importance, since in the literature to date two measures of the intensity of downside risk aversion have been proposed. First, we have the product of risk aversion and prudence, $\left(-\frac{u''}{u'}\right)\left(-\frac{u'''}{u''}\right) = \frac{u'''}{u'}$, and second we have the Schwartzian derivative, $u'''/u' - (3/2)(u''/u')^2 = (-u''/u')\left[-(u'''/u'') - (3/2)(-u''/u')\right]$. Positive prudence, which shares the same sufficient condition as does positive downside risk aversion, namely that marginal utility is convex, has also been closely linked by some authors to behavior related to downside risk aversion (see, for example, Menezes et al. (1980), Kimball (1990), and Jindapon and Nielson (2007)), but it is not true that greater prudence is equivalent to greater downside risk aversion in the traditional sense. On the other hand, an increase in $\frac{u'''}{u'}$ does lead to a greater willingness to accept the secondary risk to the downside rather than having it on the upside, as was shown by Crainich and Eeckhoudt (2007). Both of the accepted measures indicate that we cannot say anything about traditional downside risk aversion without implying the measures of both absolute risk aversion and absolute prudence.

For what follows, define:

Definition 3 *u_2 is "strongly" more prudent than u_1 if, for all w it holds that $u_2'''(w) \geq u_1'''(w)$ and $-u_2''(w) \leq -u_1''(w)$, with strict inequality for at least one of the two relationships. If both inequalities are strict, then u_2 is "strictly strongly" more prudent than u_1.*

We have the following result:

Theorem 5 *If we have two utility functions, u_1 and u_2 such that u_2 is strongly more prudent than u_1, then $\frac{1}{2} < \lambda_1^* < \lambda_2^* < 1$.*

Proof. *See Appendix 2.* ■

Note that this theorem says nothing about how the two utility functions rank in terms of the Eeckhoudt and Crainich measure of intensity of downside risk aversion, or the Swartzian derivative.

5 Conclusion

In this paper we have re-considered the definition and the comparative statics of downside risk aversion. There has been considerable debate concerning the meaning and characterisation of greater downside risk aversion. Here, rather than looking at willingness to pay or to accept (for which we find certain theoretical difficulties regarding existence), we have modelled downside risk aversion as an optimal allocation of a secondary, zero-mean, risk over the states of nature of a primary risk. Using this methodology, we find a natural definition of downside risk aversion as the choice of holding more than one-half of the secondary risk on the upside of the primary risk. We then interpret greater "allocative" downside risk aversion as a greater share of the secondary risk that is held on the upside of the primary lottery.

Is this interpretation of downside risk averison superior to the more traditional methodology in which one considers willingness to pay, or to accept, to locate all of a zero mean risk on one side or the other of a primary binomial risk? We believe so, since it is a general fact that risks can be allocated - over parties, over states of nature, etc. Indeed, the general theory of optimal risk bearing and insurance is founded on such an assumption. Since the traditional downside risk aversion methodology considers a sub-optimal allocation of the allocatable risk, we find that our approach (which corrects for this sub-optimal allocation) does indeed have a good deal of theoretical logic attached.

Besides introducing and defining allocative downside risk aversion, we have looked at some of the comparative statics of allocative downside risk aversion. We find that it is unambiguously aggrevated by an increase in the size of the primary risk in a way that looks to be related to the concept of temperence. We also find that allocative downside risk averison is increasing in the wealth and the size of the allocatable risk if the fourth derivative of utility is positive. Finally we define the concept of strongly greater prudence, and prove that a strong increase in prudence aggrevates allocative downside risk aversion.

Our research agenda on allocative downside risk aversion is still rather populated. The present study has suggested that the results presented are only the starting point of what may turn out to be a promising research path. We wonder what sort of comparative statics effects can be proved under an assumption of negative fourth derivative of utility? We also remain interested in discovering the type of utility transformation that captures a strong increase in prudence.

References

[1] Crainich, D. and L. Eeckhoudt (2007), "On the Intensity of Downside Risk Aversion", Journal of Risk and Uncertainty, 36; 267-76.

[2] Diamond, P. and J. Stiglitz (1974), "Increases in Risk and in Risk Aversion", Journal of Economic Theory, 8; 337-360.

[3] Gollier, C. and J. Pratt (1996), "Risk Vulnerability and the Tempering Effect of Background Risk", Econometrica, 64; 1109-23.

[4] Jindapon, P. and W. Neilson (2007), "Higher Order Generalizations of Arrow-Pratt and Ross Risk Aversion: A Comparative Statics Approach", Journal of Economic Theory, 136; 719-28.

[5] Kimball, M. (1990), "Precautionary Saving in the Small and in the Large", Econometrica, 58; 53-73.

[6] Keenan, D. and A. Snow (2002), "Greater Downside Risk Aversion", Journal of Risk and Uncertainty, 24; 267-77.

[7] Keenan, D. and A. Snow (2009a), "Greater Downside Risk Aversion in the Large", Journal of Economic Theory, 144; 1092-1101.

[8] Keenan, D. and A. Snow (2009b), "The Schwartzian Derivative as a Ranking of Downside Risk Aversion", working paper, University of Georgia.

[9] Menezes, C., C. Geiss and J. Tressler (1980), "Increasing Downside Risk", American Economic Review, 70; 921-32.

[10] Modica, S. and M. Scarsini (2005), "A Note on Comparative Downside Risk Aversion", Journal of Economic Theory, 122; 267-71.

Appendix 1

Define

$$\begin{aligned} U(\lambda) &= \frac{1}{2}Eu(w + x_1 + \lambda\widetilde{y}) + \frac{1}{2}Eu(w - x_2 + (1-\lambda)\widetilde{y}) \\ &= \frac{1}{2}\left[Eu(w + x_1 + \lambda\widetilde{y}) + Eu(w - x_2 + (1-\lambda)\widetilde{y})\right] \end{aligned}$$

We want to show that if marginal utility, u', is convex, then $U(0) < U(1)$. We have

$$\begin{aligned} 2U(0) &= Eu(w + x_1) + Eu(w - x_2 + \widetilde{y}) = u(w + x_1) + Eu(w - x_2 + \widetilde{y}) \\ 2U(1) &= Eu(w + x_1 + \widetilde{y}) + Eu(w - x_2) = Eu(w + x_1 + \widetilde{y}) + u(w - x_2) \end{aligned}$$

Thus we need to show that

$$u(w + x_1) + Eu(w - x_2 + \widetilde{y}) < Eu(w + x_1 + \widetilde{y}) + u(w - x_2)$$

This re-orders to

$$Eu(w - x_2 + \widetilde{y}) - u(w - x_2) < Eu(w + x_1 + \widetilde{y}) - u(w + x_1)$$

or

$$Eu(w_2 + \widetilde{y}) - u(w_2) < Eu(w_1 + \widetilde{y}) - u(w_1)$$

where $w_1 = w + x_1$ and $w_2 = w - x_2$, so that $w_1 > w_2$.

Now, define $J(w) \equiv Eu(w + \widetilde{y}) - u(w)$. What we need to show is that $J(w_1) > J(w_2)$. The slope of J is $J'(w) = Eu'(w + \widetilde{y}) - u'(w)$. Now, if u' is convex, then we know that $Eu'(w + \widetilde{y}) > u'(w + E\widetilde{y})$. Since, for the case at hand, we have $E\widetilde{y} = 0$, this now reads $Eu'(w + \widetilde{y}) > u'(w)$. However, this says that the function $J(w)$ is increasing, and so $J(w_1) > J(w_2)$ as required.

Appendix 2

In order to prove the result it is useful to re-write the problem in terms of the density and distribution of the secondary risk. Concretely, the marginal utility of λ was given above as:

$$U'(\lambda) = \frac{1}{2}Eu'(w + x_1 + \lambda\tilde{y})\tilde{y} - \frac{1}{2}Eu'(w - x_2 + (1 - \lambda)\tilde{y})\tilde{y}$$

However, assuming that the density corresponding to $\widetilde{y}$ is $f(y)$, and that this variable is defined on support (a, b), then we get

$$U'(\lambda) = \frac{1}{2}\int_a^b u'(w + x_1 + \lambda y)yf(y)dy - \frac{1}{2}\int_a^b u'(w - x_2 + (1 - \lambda)y)yf(y)dy$$

Integrating by parts, this becomes

$$U'(\lambda) = -\frac{1}{2}\int_a^b \lambda u''(w+x_1+\lambda y)H(y)dy+\frac{1}{2}\int_a^b (1-\lambda)u''(w-x_2+(1-\lambda)y)H(y)dy \tag{3}$$

where $H(y) = \int_a^y sf(s)ds$. Note that: 1) $H(a) = H(b) = 0$; 2) $H'(y) = yf(y)$, and so $H'(y) < 0$ if $y < 0$, and $H'(y) > 0$ if $y > 0$; 3) therefore it follows that $H(y) < 0$ for $y \in (a, b)$.

Now, in order to consider how a change in the utility function $u(x)$ affects the value of λ^*, we assume that utility depnds upon a shift variable α. That is, we now assume that, when wealth is z, rather than $u(z)$ utility is $u(\alpha, z)$. In this way a value of α defines a family, or class, of utility functions (see Diamond and Stiglitz 197?). Given this, we now get $\lambda = \lambda^*(\alpha)$, and we can consider how a change in α affects the optimal solution.

Using standard notation for derivatives, the optimal solution $\lambda^*(\alpha)$ is the solution to:

$$\int_a^b u_x(\alpha, w + x_1 + \lambda y)yf(y)dy - \int_a^b u_x(\alpha, w - x_2 + (1 - \lambda)y)yf(y)dy = 0$$

Taking the derivative with respect to α gives:

$$\int_a^b u_{\alpha x}(\alpha, w+x_1+\lambda y)yf(y)dy + \int_a^b u_{xx}(\alpha, w+x_1+\lambda y)\lambda_\alpha y^2 f(y)dy -$$
$$-\int_a^b u_{\alpha x}(\alpha, w-x_2+(1-\lambda)y)yf(y)dy +$$
$$+\int_a^b u_{xx}(\alpha, w-x_2+(1-\lambda)y)\lambda_\alpha y^2 f(y)dy = 0$$

Solving for $\lambda_\alpha = \frac{\partial \lambda^*}{\partial \alpha}$ we find that:

$$\frac{\partial \lambda^*}{\partial \alpha} = -\frac{\int_a^b u_{\alpha x}(\alpha, w+x_1+\lambda y)yf(y)dy - \int_a^b u_{\alpha x}(\alpha, w-x_2+(1-\lambda)y)yf(y)dy}{\int_a^b u_{xx}(\alpha, w+x_1+\lambda y)y^2 f(y)dy + \int_a^b u_{xx}(\alpha, w-x_2+(1-\lambda)y)y^2 f(y)dy}$$

the denominator of which satisfies:

$$D_2(\alpha,\lambda) = \int_a^b [u_{xx}(\alpha, w+x_1+\lambda y) + u_{xx}(\alpha, w-x_2+(1-\lambda)y)]y^2 f(y)dy < 0$$

Thus, the sign of $\frac{\partial \lambda^*}{\partial \alpha}$ is the same as the sign of:

$$\begin{aligned} D_1(\alpha,\lambda) &= \int_a^b [u_{\alpha x}(\alpha, w+x_1+\lambda y) - u_{\alpha x}(\alpha, w-x_2+(1-\lambda)y)]yf(y)dy = \\ &= -\int_a^b [\lambda u_{\alpha xx}(\alpha, w+x_1+\lambda y) - \\ &\quad -(1-\lambda)u_{\alpha xx}(\alpha, w-x_2+(1-\lambda)y)]H(y)dy \end{aligned}$$

where the second line is found by deriving by parts. Now, using the second expression for $D_1(\alpha,\lambda)$ we get (recall that $H(y) < 0$, and of course $w+x_1+\frac{1}{2}y > w-x_2+\frac{1}{2}y$ for all y):

$$\begin{aligned} D_1(\alpha,\frac{1}{2}) &= -\frac{1}{2}\int_a^b [u_{\alpha xx}(\alpha, w+x_1+\frac{1}{2}y) - u_{\alpha xx}(\alpha, w-x_2+\frac{1}{2}y)]H(y)dy \geq 0 \\ if\ u_{\alpha xxx} &\geq 0. \end{aligned}$$

And using the first expression for $D_1(\alpha,\lambda)$ we get:

$$\begin{aligned} \frac{\partial}{\partial \lambda}D_1(\alpha,\lambda) &= \int_a^b [u_{\alpha xx}(\alpha, w+x_1+\lambda y) + u_{\alpha xx}(\alpha, w-x_2+(1-\lambda)y)]y^2 f(y)dy \geq 0 \\ if\ u_{\alpha xx} &\geq 0. \end{aligned}$$

Therefore, if $u_{\alpha xx} \geq 0$ (a family of utility with u_{xx} non-decreasing in α) and $u_{\alpha xxx} \geq 0$ (a family of utility with u_{xxx} non-decreasing in α) it turns out that $D_1(\alpha,\lambda)$ is non-decreasing in λ. And since $D_1(\alpha,\frac{1}{2}) \geq 0$, it must hold that $D_1(\alpha,\lambda) \geq 0$ for all $\lambda > \frac{1}{2}$. In particular, if at least one of the inequalities

$u_{\alpha xx} \geq 0$ and $u_{\alpha xxx} \geq 0$, then it follows that $D_1(\alpha, \lambda) > 0$ for all $\lambda > \frac{1}{2}$. In such a case we can conclude that:

$$\frac{\partial \lambda^*}{\partial \alpha} = -\frac{D_1(\alpha, \lambda^*)}{D_2(\alpha, \lambda^*)} > 0$$

This proves that, under the assumptions of $u'' < 0$ and $u''' > 0$, changes in the utility function such that $u_2''' \geq u_1'''$ and $u_1'' \leq u_2''$, with strict inequality for at least one of these relationships, then $\frac{1}{2} < \lambda_1^* < \lambda_2^* < 1$. But, this is what we have defined as a strong increase in prudence.