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A New Existence and Uniqueness Theorem for Continuous Games

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A New Existence and Uniqueness Theorem for Continuous Games*

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Abstract:

This paper derives a general sufficient condition for existence and uniqueness in continuous games using a variant of the contraction mapping theorem applied to mapping from a subset of the real line on to itself. We first prove this contraction mapping variant, and then show how the existence of a unique equilibrium in the general game can be shown by proving the existence of a unique equilibrium in an iterative sequence of games involving such $\mathbb{R}$-to-$\mathbb{R}$ mappings. Finally, we show how a general condition for this to occur is that a matrix derived from the Jacobean matrix of best-response functions be have positive leading principal minors, and how this condition generalises some existing uniqueness theorems for particular games.

Key Words: Existence, Uniqueness, Continuous Games, Contraction Mapping Theorem

JEL Classifications: C62, C72, D43:

A New Existence and Uniqueness Theorem for Continuous Games

1. Introduction:

Many oligopoly models fall into the class of continuous games in which each player chooses a strategy from a connected subset of the real line. For example, in Cournot models, each firm chooses a quantity to produce, in product differentiation models firms typically choose a price, and so on.

Conditional on showing existence, there are a number of conditions that have been found for particular subsets of this general class of models, which can be used to prove uniqueness of the equilibrium.[1] Generally, these involve a trade-off between generality and ease of application. For instance, the contraction mapping theorem offers a general sufficient condition for uniqueness, but for it to be useable, one needs to show that the particular mapping for which a fixed point defines an equilibrium constitutes a contraction mapping, a task that is not always straightforward. At the opposite end of the generality/usability continuum, are uniqueness conditions that are specific to particular applications. For instance, conditions for a unique equilibrium in the Cournot quantity-setting oligopoly model have been derived by Szidarovszky and Yakowitz (1977), Gaudet and Salant (1991), and Long and Soubeyran (2000).

Between these two extremes, there are a number of general uniqueness conditions that can be expressed in terms of the signs of the principal minors of a matrix derived from the Jacobean matrix of best-response functions. These include results derived from the Gale and Nikaido (1965) theorem on univalent mappings, and results making use of the Poincare-Hopf index theorem. (See, for example, Simsek, Ozdaglar, and Acemoglu, 2007.)

A limitation of these uniqueness results, however, is that they depend on a prior demonstration of existence, and that, to the extent that existence is proved by use of the

[1] A good survey of existing uniqueness theorems is contained in Cachon and Netessine (2004).

Brouwer or Kakutani fixed-point theorems, this implies a requirement that the strategy space be bounded.

For this reason, it is desirable to find easy-to-apply conditions under which the contraction-mapping theorem can be used to show uniqueness, since the contraction mapping theorem provides a quite general condition for there to be a unique fixed point, with the added benefit that it guarantees existence without the requirements that the space being mapped onto itself be convex or bounded.

In this paper, we derive a variant of the contraction mapping theorem, and present an approach in which this variant is applied iteratively to generate a new fixed-point theorem, which, like the contraction mapping theorem, implies both existence and uniqueness without the requirement that the strategy space be bounded. The conditions for this theorem can, like the univalence and index-theorem results referred to above, be expressed in terms of this signs of the principal minors of a matrix derived from the Jacobean matrix of best-response functions.

The contribution of this paper is threefold: First, the theorem derived in this paper offers a slight generalisation of the *P*-matrix results derived from the Gale-Nikkaido theorem or index theory in the case where the strategy-space is bounded; second, and most important, by showing existence in cases where the strategy space is unbounded, it extends those existing uniqueness results to the unbounded case; and third, the derivation does not rely on concepts from differential topology, and so is perhaps more accessible than those derived from index theory.

In the next section, we present the contraction mapping theorem and a related, less-restrictive theorem for the special case of a mapping from a subset of the real line onto itself. Section 3 lays out the general problem and shows by example why the conventional characterisation of an equilibrium as a fixed point of an $\mathbb{R}^n$-to-$\mathbb{R}^n$ mapping is too restrictive. Sections 4 shows how an equilibrium in the general game can be defined in terms of a sequence of contraction mappings involving $\mathbb{R}^1$-to-$\mathbb{R}^1$ mappings; Section 5 then shows how the existence and uniqueness condition derived iteratively in this way can be represented in terms of the slopes of the best-response functions of each player. Section 6 shows how this general condition encompasses and generalises many existing results. Section 7 concludes.

2. The Contraction Mapping Theorem and a Related Result.

A. The Contraction Mapping Theorem in Euclidean Space.

Typically in oligopoly models, the existence of an equilibrium is proved by showing the existence of a fixed point in a mapping from a subset of Euclidean space onto itself. Let $\mathbb{X}$ be a subset of $\mathbb{R}^n$, and let $f:\mathbb{X}\mapsto\mathbb{X}$ be a single-valued function mapping $\mathbb{X}$ onto itself. In this context, the definition of a contraction mapping and the contraction mapping theorem are as follows:

Definition 1:

If there exists $\beta\in(0,1)$ and a norm $\|\boldsymbol{x}\|$ such that

$$\|f(\boldsymbol{y})-f(\boldsymbol{x})\|\le\beta\|\boldsymbol{y}-\boldsymbol{x}\| \qquad \forall\boldsymbol{x},\boldsymbol{y}\in\mathbb{X}, \tag{1}$$

then *f* is a *contraction mapping*.

Theorem 1 (The Contraction Mapping Theorem):

If $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$ and *f* is a contraction mapping, then

a) (existence and uniqueness) there exists a unique fixed point $x^*\in\mathbb{X}$ such that $f(x^*)=x^*$,

b) (convergence) for any $x\in\mathbb{X}$ and $n\ge 1$, $\|f^n(x)-x^*\|\le\beta^n\|x-x^*\|$.

The contraction mapping theorem has three advantages over the Brouwer or Kakutani fixed point-theorems if a contraction mapping can be shown to exist: First, and most importantly, it shows uniqueness as well as existence; second, it does not require that the set $\mathbb{X}$ be bounded; and third, it has the convergence property.

The convergence property implies that the unique fixed point can easily be found numerically. It can be useful in a game-theoretic context if we imagine the Nash equilibrium to be one iteration of a repeated game, as it suggests that the Nash-equilibrium outcome can be stable in the sense that if every period each player chooses the best response to the previous-period strategies of the other players, the game will converge to the unique equilibrium.

Such dynamic interpretations of a static equilibrium are not always appropriate, however, and numerical solveability is rarely important. If we only require existence and uniqueness and not convergence, we can, in principle, relax Condition (1). We do this below for the case of case of $\mathbb{R}^1$-to-$\mathbb{R}^1$ mappings.

B. The Contraction Mapping Theorem in $\mathbb{R}^1$ *Space.*

In this paper, we show how existence of an equilibrium that is a point in Euclidean *n* space, can be represented as a set of fixed points of a sequence of mappings from the real line onto itself. For $\mathbb{R}^1$-to-$\mathbb{R}^1$ mappings, the natural norm to use is the absolute value, $\|x\| = |x|$, and the definition of a contraction mapping becomes as follows:

Definition 2:

If there exists $\beta \in (0,1)$ such that

$$\frac{|f(y) - f(x)|}{|y - x|} \le \beta \qquad \forall x, y \in \mathbb{X}, \tag{2}$$

then *f* is a *contraction mapping*.

In words, this says that the straight line between any two points on the graph of the function, must have a slope in the interval (-1,1). If, we don’t require the convergence property, we only require that the slope be less than 1 and that the function be continuous. We will define such a function as a “quasi-contraction mapping”.

Definition 3:

If there exists $\beta \in (0,1)$ such that

$$\frac{f(x)-f(y)}{x-y} \le \beta \qquad \forall x, y \in \mathbb{X}, \tag{3}$$

and if *f* is continuous on $\mathbb{X}$, then *f* is a *quasi-contraction mapping*.

This gives us the following variant of the contraction mapping theorem:

Theorem 2:

If $\mathbb{X}$ is a closed, connected subset of $\mathbb{R}$ and *f* is a quasi-contraction mapping, then there exists a unique fixed point, $x^* \in \mathbb{X}$, such that $f(x^*) = x^*$.

Proof:

First we show that a fixed point must exist. For any $x_0 \in \mathbb{X}$, we have $f(x_0) \ge x_0$, or $f(x_0) \le x_0$. If $f(x_0) \ge x_0$, then define,

$$y_0 = \begin{cases} \max\{x \mid x \in \mathbb{X}\} & \text{if } \mathbb{X} \text{ is bounded above} \\ \dfrac{f(x_0)-\beta x_0}{1-\beta} & \text{otherwise} \end{cases}. \tag{4}$$

If $\mathbb{X}$ is not bounded above, we have, from (3) and (4),

$$\begin{aligned} & f(y_0) - f(x_0) \le \beta(y_0 - x_0) \\ \Rightarrow \quad & f(y_0) \le \beta y_0 + f(x_0) - \beta x_0 \\ \Rightarrow \quad & f(y_0) \le y_0. \end{aligned}$$

If $\mathbb{X}$ is not bounded above, we have directly that

$$f(y_0) \le y_0.$$

By the intermediate value theorem, therefore, there exists $x^* \in [x_0, y_0]$ such that $f(x^*) = x^*$. A similar argument holds if $f(x_0) \le x_0$.

To show uniqueness, let $x^* \in \mathbb{X}$ be a fixed point of *f*. Then $\forall x > x^*$,

$$\frac{f(x)-f(x^*)}{x-x^*} \leq \beta < 1$$

$$\Rightarrow \quad f(x)-f(x^*) \leq \beta(x-x^*) < x-x^*$$

$$\Rightarrow \quad f(x)-x < f(x^*)-x^* = 0.$$

Thus any fixed point of *f* must be the maximum fixed point, implying that only one can exist.

□

Finally, if, in addition to the above assumptions, we assume that *f* is differentiable almost everywhere, then Condition (2) is equivalent to the following:

$$\begin{array}{l} \text{a)} \quad f \text{ is continuous over } \mathbb{X}, \\ \text{b)} \quad |f'(x)| \leq \beta \quad \forall x \in \mathbb{X} \text{ where } f \text{ is differentiable.} \end{array} \tag{5}$$

Similarly, Condition (3) is equivalent to the following:

$$\begin{array}{l} \text{a)} \quad f \text{ is continuous over } \mathbb{X}, \\ \text{b)} \quad f'(x) \leq \beta \quad \forall x \in \mathbb{X} \text{ where } f \text{ is differentiable.} \end{array} \tag{6}$$

This gives a general uniqueness theorem that we shall use in this paper:

Theorem 3:

Let $\mathbb{X}$ be a closed, connected subset of $\mathbb{R}$, and let *f* be a single-valued continuous function from $\mathbb{X}$ onto itself that is differentiable almost everywhere. Then if, for some $\beta \in (0,1)$

a) $f'(x) < \beta \qquad \forall x \in \mathbb{X}$ where f is differentiable,

there exists a unique fixed point $x^* \in \mathbb{X}$ such that $f(x^*) = x^*$.

If, in addition, we have

b) $f'(x) > -\beta \qquad \forall x \in \mathbb{X}$ where f is differentiable,

then the fixed point is stable in the sense that for any $x \in \mathbb{X}$ and $n \geq 1$, $|f^n(x)-x^*| \leq \beta^n |x-x^*|$.

In the next section we show how the contraction mapping theorem in $\mathbb{R}^n$ space is used to establish uniqueness in the class of games considered in this paper, and show by example why we seek to reduce the problem to one involving $\mathbb{R}^1$-to-$\mathbb{R}^1$ mappings.

3. The General Problem:

A. Notation:

Imagine that there are *n* players. We employ the following notation. The strategy space for each player *i* is $\mathbb{X}_i$, and the set of all possible combinations of strategies for all players is $\mathbb{X}^n$. Let $x_i \in \mathbb{X}_i$ denote a strategy for player *i* and let $\boldsymbol{x} = (x_1, ... x_n) \in \mathbb{X}^n$ denote a strategy combination for all players. Rather than specify the payoffs for each player, we will express everything in terms of the best-response functions. Specifically, let $f_i(\boldsymbol{x})$ be the best-response of player *i* to the combination of strategies of the other players.[2]

We impose the following restrictions on this general set-up:

Assumption 1:

a) For each *i*, $\mathbb{X}_i$ is a connected subset of the real line;

b) for each *i*, f_i is continuous, single-valued, and differentiable almost everywhere over $\mathbb{X}^n$.

We do not require that the f_i be fully differentiable so that the model will be able to handle non-differentiabilities that can arise from boundary solutions to an individual player's optimisation problem. For ease of exposition, however, when presenting

2 It would be more conventional to define f_i over $\mathbb{X}^{n-1}$ rather than $\mathbb{X}^n$, writing the best-response as $f_i(\boldsymbol{x}_{-i})$, where $\boldsymbol{x}_{-i}$ is the vector of strategies of the players other than player *i*. We use the more general notation here so that the result derived in Section 4 is a more general fixed-point theorem, and not confined to the game-theoretic interpretation. The game-theoretic application is a special case in which $\partial f_i / \partial x_i = 0 \ \forall i$.

expressions involving derivatives we will omit the repeated caveat, "$\forall \boldsymbol{x} \in \mathbb{X}^n$ where f_i is differentiable", but this is implied.

Proofs of existence of an equilibrium in this class of games typically proceed by defining the aggregate best-response function, $\boldsymbol{f}: \mathbb{X}^n \mapsto \mathbb{X}^n$ where $\boldsymbol{f} = (f_1, f_2, \ldots f_n)$, so that a Nash equilibrium in pure strategies is a fixed point of $\boldsymbol{f}$ and vice versa, and then appealing to the Brouwer fixed-point theorem. We want to find sufficient conditions for $\boldsymbol{f}$ to have a unique fixed point. For this, it would be sufficient to show that $\boldsymbol{f}$ is a contraction mapping. This, however, would be too restrictive, as illustrated by the following simple example.

B. A Numerical Example:

Consider a Cournot game in which the *n* players are firms choosing the quantity to produce taking the quantity produced by each of the other *n*-1 players as given. Assume that the market inverse demand curve is linear, and that each firm has a constant marginal cost of production. The linear structure satisfies the conditions required for uniqueness by Szidarovszky and Yakowitz (1977) amongst others.

To see whether the aggregate best-response function constitutes a contraction mapping, we need to define a norm. Rather than choosing a particular norm, we will consider any *p*-norm of the form

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} \left(|x_i|\right)^p \right)^{\frac{1}{p}} \text{ for some real number } p \geq 1.$$

Now consider some initial vector of outputs, $\boldsymbol{x} = (x_1, \ldots x_n)$, and a second vector, $\boldsymbol{y} = (y_1, \ldots y_n)$, where all outputs have been perturbed by the same constant, δ, so that $y_i = x_i + \delta \;\; \forall i.$ The *p*-norm for this perturbation is

$$\|\boldsymbol{y} - \boldsymbol{x}\|_p = n^{1/p}\delta.$$

The linear Cournot game produces linear best-response functions in which, at interior solutions, $\partial x_i / \partial x_j = -0.5 \quad \forall i, j \neq i.$ We therefore have $f_i(\boldsymbol{y}) - f_i(\boldsymbol{x}) = -0.5(n-1)\delta,$ and hence

$$\left\| \boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) \right\|_p = n^{1/p} 0.5(n-1)\delta.$$

For any n>2, therefore, we have

$$\left\| \boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) \right\|_p \geq \left\| \boldsymbol{y} - \boldsymbol{x} \right\|_p$$

and hence $\boldsymbol{f}$ is not a contraction mapping for any p-norm.

What this example shows is that if we wish to use the contraction mapping theorem to show existence and uniqueness in a class of models that encompasses this standard example we will need to use a different function than $\boldsymbol{f}$ for which a fixed point defines an equilibrium. In the next section, we develop an approach that enables us to transform the problem so that an equilibrium is a fixed point in a mapping from a subset of the real line onto itself.

4. An Alternative Approach:

To transform the problem, we define an equilibrium iteratively, starting with one or two players and then progressively adding more in. The procedure we follow here will derive a general fixed-point theorem for a function, $\boldsymbol{f} : \mathbb{X}^n \mapsto \mathbb{X}^n$, where $\mathbb{X}$ is a connected subset of the real line. Because we are interested in the game-theoretic application, however, we will continue to refer to the elements from the vector, $\boldsymbol{x}$, as strategies, and to fixed points as "equilibria".

Define an m-equilibrium as a set of x_i such that

$$x_i = f_i(\boldsymbol{x}_{-i}) \quad \forall i = 1..m,$$

where $m \leq n.$ That is, it is a set of strategies such that the first m players' strategies are the best response to the strategies of *all* other players, but the remaining n-m players' strategies are unconstrained.

Let $\boldsymbol{x}^m$ be the vector of strategies by the first *m* players, and let $\boldsymbol{x}^{-m}$ be the strategies of the remaining *n*-*m* players when $m < n.$ An *m*-equilibrium when $m < n$ is therefore a Nash equilibrium in $\boldsymbol{x}^m$ taking $\boldsymbol{x}^{-m}$ as given, and the *m*-equilibrium when $m = n$ is a Nash equilibrium of the full game.

Let $h_i^m(\boldsymbol{x}^{-m})$ denote an *m*-equilibrium value of x_i when $m < n$ and let $\boldsymbol{h}^m(\boldsymbol{x}^{-m})$ be an *m*-vector of those values. If there exists a unique *m*-equilibrium for each value of $\boldsymbol{x}^{-m}$, then the function $\boldsymbol{h}^m$ is single valued and defined for all $\boldsymbol{x}^{-m}$.

Our approach to finding sufficient conditions for a unique equilibrium is to find the conditions for a unique 1-equilibrium that holds for all values of $\boldsymbol{x}^{-1}$ and then to extend that by induction by finding conditions for there to exist a unique (*m*+1)-equilibrium that holds for all values of $\boldsymbol{x}^{-(m+1)}$ conditional on there being a unique *m*-equilibrium.

We will define a set of mappings, $g_m : \mathbb{X} \mapsto \mathbb{X},$ that will relate a player's strategy to itself. For *m*=1, we simply define,

$$g_1(x_1; \boldsymbol{x}^{-1}) \equiv f_1(x_1, \boldsymbol{x}^{-1}). \tag{7}$$

There is an equivalence between a 1-equilibrium and a fixed point of g_1. From Theorem 2, a sufficient condition for there to exist a unique 1-equilibrium is that g_1 be a quasi-contraction mapping. When f_1 is a best-response function, $\partial g_1 / \partial x_1 = 0$ and so this sufficient condition will always hold. In the more general case, we need $\partial g_1 / \partial x_1 < 1$.

Now imagine that there exists a unique (*m*-1)-equilibrium for each value of $\boldsymbol{x}^{-(m-1)}$. In this case, define g_m as

$$g_m(x_m; \boldsymbol{x}^{-m}) \equiv f_m(\boldsymbol{h}^{m-1}(x_m, \boldsymbol{x}^{-m}), x_m, \boldsymbol{x}^{-m}). \tag{8}$$

Again, there is an equivalence between an *m*-equilibrium and a fixed point of g_m, and so a sufficient condition for there to exist a unique *m*-equilibrium given $\boldsymbol{x}^{-m}$ is that g_m be a quasi-contraction mapping.

Definition 4:

We say that $\boldsymbol{f}$ exhibits an "iterative quasi-contraction mapping" if g_m exists and is a quasi-contraction mapping for each $m \in \{2...n\}$, and that it exhibits an "iterative contraction mapping" if g_m exists and is a contraction mapping for each $m \in \{2...n\}$.

The main result of this paper is then

Theorem 4:

If $\boldsymbol{f}$ exhibits an iterative quasi-contraction mapping, then $\boldsymbol{f}$ has a unique fixed point, and hence the game has a unique Nash equilibrium.

Proof:

As we have shown, if g_1 is a quasi-contraction mapping, there exists a unique 1-equilibrium for all values of $\boldsymbol{x}^{-1}$. If there exists a unique (*m*-1)-equilibrium for all values of $\boldsymbol{x}^{-(m-1)}$ then g_m exists, and if g_m is a quasi-contraction mapping, there exists a unique *m*-equilibrium. By induction, then, if g_m is a quasi-contraction mapping for each $m \in \{1...n\}$, then there must exist a unique *m*-equilibrium for each *m*, and hence a unique equilibrium for the full game.

□

Now imagine that each of the g_m is a full contraction mapping so that repeated applications of g_m will generate convergence to the unique fixed point. This does not imply that the full equilibrium would be stable in the way it would be if $\boldsymbol{f}$ were a contraction mapping.[3] It does, however, that imply a numerical solvability using the best-response functions in the following sense. First, note that the 2-equilibrium can found

[3] This can be seen from the Cournot example in Section 3, for which the equilibrium is not stable, but for which, as will be shown in Section 6, each of the g_m is a contraction mapping.

iteratively by alternately adjusting player 1's strategy to that of player 2 and vice versa. Then, if the m-equilibrium is iteratively solvable by sequentially adjusting each of the first m player's strategies to be on his best-response functions, and if g_{m+1} is a contraction mapping, then the (m+1)-equilibrium is iteratively solvable by adjusting the m-equilibrium to x_{m+1} and then x_{m+1} to the $\boldsymbol{h}^m$ and so on. Iterative solveability is perhaps not the most useful property one might desire of an equilibrium, but it is essentially a free result.

5. Sufficient Conditions with Calculus.

The analysis of the previous section gives sufficient conditions for uniqueness and iterative solvability that derive from our sequential approach. They are not, however, particularly user friendly. For that, we would like to express the conditions in terms of the slopes of the best-response functions.

To do this let $\boldsymbol{J}_n(\boldsymbol{x})$ be the nxn Jacobean matrix of $\boldsymbol{f}$ evaluated at $\boldsymbol{x}$, with elements $J_{ij}(\boldsymbol{x})$, so that

$$J_{ij}(\boldsymbol{x}) = \frac{\partial f_i(\boldsymbol{x})}{\partial x_j}, \qquad \forall i, j,$$

and let $\boldsymbol{A}_n(\boldsymbol{x})$ be the nxn matrix, $\boldsymbol{A}_n(\boldsymbol{x}) = \boldsymbol{I}_n - \boldsymbol{J}_n(\boldsymbol{x})$, where $\boldsymbol{I}_n$ is the nxn identity matrix, with the elements of $\boldsymbol{A}_n(\boldsymbol{x})$ denoted, $a_{ij}(\boldsymbol{x})$. Finally, let $\boldsymbol{J}_m(\boldsymbol{x})$ and $\boldsymbol{A}_m(\boldsymbol{x})$ be the mxm submatrices comprising the first m rows and first m columns of $\boldsymbol{J}_n(\boldsymbol{x})$ and $\boldsymbol{A}_n(\boldsymbol{x})$, respectively. The derivatives of the functions g_m can be expressed in terms of the determinants of the $\boldsymbol{A}_m(\boldsymbol{x})$ as follows:

Theorem 5:

If g_m exists, then

$$\frac{dg_1(\boldsymbol{x})}{dx_1} = 1 - \left|\boldsymbol{A}_1(\boldsymbol{x})\right|, \text{ and} \tag{9}$$

$$\frac{dg_m(\boldsymbol{x})}{dx_m} = 1 - \frac{|\boldsymbol{A}_m(\boldsymbol{x})|}{|\boldsymbol{A}_{m-1}(\boldsymbol{x})|} \text{ for } m \geq 2. \tag{10}$$

Proof:

Note that

$$|\boldsymbol{A}_1(\boldsymbol{x})| = a_{11}(\boldsymbol{x}) = 1 - \frac{\partial g_1(\boldsymbol{x})}{\partial x_1},$$

so Equation (9) holds trivially.

For $m \geq 2,$ we have from Equation (8) that

$$\frac{dg_m}{dx_m} = \sum_{i=1}^{m-1} \frac{\partial f_m}{\partial x_i} \cdot \frac{\partial h_i^{m-1}}{\partial x_m} + \frac{\partial f_m}{\partial x_m}. \tag{11}$$

Define $\boldsymbol{A}_{Cm}(\boldsymbol{x})$ as the column vector containing the first *m*-1 elements of the *m*'th column of $\boldsymbol{A}_n(\boldsymbol{x}),$ and define $\boldsymbol{A}_{Rm}(\boldsymbol{x})$ similarly as the row vector containing the first *m*-1 elements of the *m*'th row of $\boldsymbol{A}_n(\boldsymbol{x}),$ so that $\boldsymbol{A}_m(\boldsymbol{x}),$ is the partitioned matrix

$$\boldsymbol{A}_m(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{A}_{m-1}(\boldsymbol{x}) & \boldsymbol{A}_{Cm}(\boldsymbol{x}) \\ \boldsymbol{A}_{Rm}(\boldsymbol{x}) & a_{mm}(\boldsymbol{x}) \end{bmatrix}.$$

We can then rewrite Equation (11) as

$$\frac{dg_m}{dx_m} = -\boldsymbol{A}_{Rm}(\boldsymbol{x}) \frac{\partial \boldsymbol{h}^{m-1}}{\partial x_m} + \frac{\partial f_m}{\partial x_m}. \tag{12}$$

The h_i^{m-1} variables are defined by the fixed point in the *m*-equilibrium

$$h_i^{m-1} \equiv f_i(\boldsymbol{h}_{-i}^{m-1}, x_m, \boldsymbol{x}^{-m}) \qquad \forall i \leq m-1.$$

Total differentiation yields

$$\frac{\partial h_i^{m-1}}{\partial x_m} = \sum_{j=1}^{m-1} \frac{\partial f_i}{\partial h_j} \cdot \frac{\partial h_j^{m-1}}{\partial x_m} + \frac{\partial f_i}{\partial x_m}.$$

In matrix notation this gives

$$\frac{\partial \boldsymbol{h}^{m-1}}{\partial x_m} = \boldsymbol{J}_m(\boldsymbol{x})\frac{\partial \boldsymbol{h}^{m-1}}{\partial x_m} - \boldsymbol{A}_{Cm}(\boldsymbol{x})$$

$$\Rightarrow \quad \frac{\partial \boldsymbol{h}^{m-1}}{\partial x_m} = -\boldsymbol{A}_{m\text{-}1}^{-1}(\boldsymbol{x})\boldsymbol{A}_{Cm}(\boldsymbol{x})$$

so Equation (12) becomes

$$\frac{dg_m}{dx_m} = \boldsymbol{A}_{Rm}(\boldsymbol{x})\boldsymbol{A}_{m\text{-}1}^{-1}(\boldsymbol{x})\boldsymbol{A}_{Cm}(\boldsymbol{x}) + \frac{\partial f_m}{\partial x_m}.$$

Finally, note that[4]

$$\boldsymbol{A}_{Rm}(\boldsymbol{x})\boldsymbol{A}_{m\text{-}1}^{-1}(\boldsymbol{x})\boldsymbol{A}_{Cm}(\boldsymbol{x}) = a_{mm} - \frac{|\boldsymbol{A}_m(\boldsymbol{x})|}{|\boldsymbol{A}_{m-1}(\boldsymbol{x})|}, \tag{13}$$

and that

$$\frac{\partial f_m}{\partial x_m} = 1 - a_{mm},$$

so that

$$\frac{dg_m(\boldsymbol{x})}{dx_m} = 1 - \frac{|\boldsymbol{A}_m(\boldsymbol{x})|}{|\boldsymbol{A}_{m-1}(\boldsymbol{x})|}.$$

□

The conditions for an iterative quasi-contraction mapping can now be stated in terms of the determinants of the $\boldsymbol{A}_m(\boldsymbol{x})$ matrices:

Theorem 6:

a) If there is an ordering of players, indexed by 1..*n*, such that for some $\varepsilon \in (0,1)$

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[4] Equation (13) is a special case of the general result for partitioned matrices that

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{B} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}||\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{C}|$$

for square matrices ***A*** and ***D***, where ***A*** is non-singular. For a proof of this result, see, for example, Rao (1965, p28).

$$\left|\boldsymbol{A}_m(\boldsymbol{x})\right| \geq \varepsilon > 0 \; \forall m \in \{2..n\} \tag{14}$$

then the best-response function exhibits an iterative quasi-contraction mapping.

b) If, in addition, we have

$$\frac{\left|\boldsymbol{A}_m(\boldsymbol{x})\right|}{\left|\boldsymbol{A}_{m-1}(\boldsymbol{x})\right|} \leq 2-\varepsilon < 2 \quad \forall m \in \{2..n\}. \tag{15}$$

then the best-response function exhibits an iterative quasi-contraction mapping.

Proof:

Follows automatically from Theorems 3 and 4.

□

Condition (14) gives the general existence and uniqueness condition of this paper—that the function $\boldsymbol{f}$ has a unique fixed point if the leading principal minors of $(\boldsymbol{I}_n(\boldsymbol{x}) - \boldsymbol{J}_n(\boldsymbol{x}))$ are positive for all $\boldsymbol{x}$, and bounded away from zero.

6. Relationship to Other Existence and Uniqueness Conditions.

In this paper, we have found a condition under which there exists a unique fixed point for the single-valued function, $\boldsymbol{f}: \mathbb{X}^n \mapsto \mathbb{X}^n$, where $\mathbb{X}$ is a closed, connected subset of the real line and $\boldsymbol{f}$ is continuous. Note that if $\mathbb{X}$ is bounded, the existence of a fixed point is guaranteed by the Brouwer fixed-point theorem and so Theorem 6 is a primarily a uniqueness theorem.

Typically, existence theorems for pure-strategy Nash equilibria that deal with more general games, do require bounded strategy spaces. This includes games where the best-response functions are not necessarily single valued that make use of the Kakutani fixed-point theorem, and the existence theorem for supermodular games in which the best-response functions are not necessarily continuous, which makes use of the Tarski fixed-point theorem. (See Fudenberg and Tirole (1991) for a description of supermodular games.)

The contraction mapping theorem is probably the most important general existence theorem for games with unbounded strategy spaces. Theorem 6 uses a weaker requirement than a full contraction mapping to establish both existence and uniqueness, and provides a simple Jacobean representation of that condition. Interestingly, this matrix representation of the condition, is very similar to a number of existing uniqueness proofs that require a prior demonstration of existence. The major contribution of Theorem 6, therefore, is to extend those results to the case of unbounded strategy spaces.

In the remainder of this section, we survey those existing uniqueness conditions.

A. Univalent Mapping Theorems:

Two of the best known papers providing a generic set of sufficient conditions for uniqueness in games of the form analysed here are Gale and Nikaido (1965) and Rosen (1965).

Although the derivation is very different, Gale and Nikaido's sufficient condition is very similar to Theorem 6. Their condition, which applies when the strategy-space, $\mathbb{X}$ is bounded, is that $(\boldsymbol{I}_n - \boldsymbol{J}_n(\boldsymbol{x}))$ be a *P*-matrix—that is, that all principal minors be positive. For bounded strategy spaces, Theorem 6 is slightly more general, in that it only requires that the *leading* principal minors be positive.[5] More importantly, Theorem 6 extends the Gale-Nikaido condition to the case of unbounded strategy spaces by proving existence in those cases.

Rosen considers a very general game structure in which the strategy space for any player can be conditional on the strategy chosen by another (as could happen in a coalition game). In the special case, however, where the player's strategy spaces are orthogonal to each other, i.e. the class of games considered in this paper, Rosen's sufficient condition can be written as follows:

[5] The Gale-Nikaido theorem only requires that the principal minors be positive rather than bounded away from zero. Since the theorem only applies to equilibria contained within a closed rectangular region, however, the requirement that all principal minors be positive is the equivalent to requiring that they be bounded above zero.

If there exists a diagonal matrix ***R****, with diagonal terms* $r_{ii} > 0 \ \forall i$ *such that the symmetric matrix* $(\boldsymbol{RA}) + (\boldsymbol{RA})'$ *is positive definite, then there is a unique equilibrium.*

The main result of this paper generalises this result in two ways: First, Rosen establishes existence by means of the Kakutani fixed-point theorem, and thus again requires each player's strategy space be bounded; second, Rosen's sufficient condition is strictly encompassed by the conditions of Theorem 6 here, as shown by the following result.

Theorem 7:

For any symmetric nxn matrix, ***A****, if there exists a diagonal matrix* ***R****, with diagonal terms* $r_{ii} > 0 \ \forall i$ *such that the symmetric matrix* $(\boldsymbol{RA}) + (\boldsymbol{RA})'$ *is positive definite, then the leading principal minors of* ***A*** *will be positive, but the reverse is not necessarily true.*

Proof:

$(\boldsymbol{RA}) + (\boldsymbol{RA})'$ is positive definite if and only if $\boldsymbol{RA}$ is positive definite, which implies that the principal minors of $\boldsymbol{RA}$ are all positive, and hence that the principal minors of ***A*** are positive. The fact that a non-symmetric matrix with positive principal minors is not necessarily positive definite, however, allows one to construct counterexamples in which the conditions for Theorem 6 are met, but not for Rosen's theorem. For one such counterexample, consider the following matrix.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 1 & 0 \\ 0 & .5 & 1 \end{bmatrix}.$$

A has positive principal minors. Without loss of generality, we can set $r_1 = 1,$ so that

$$(\boldsymbol{RA}) + (\boldsymbol{RA})' = \begin{bmatrix} 2 & 2r_2 & 8 \\ 2r_2 & 2r_2 & .5r_3 \\ 8 & .5r_3 & 2r_3 \end{bmatrix}.$$

For the second principal minor to be positive, we need $r_2 < 1.$ It is easy to show that the determinant of the full matrix is concave in r_3 given $r_2,$ and hence that the determinant-maximising value of r_3 given r_2 is

$$r_3 = 8(3 - r_2)r_2.$$

Substituting in this value of $r_3,$ it is trivial to show that the determinant of the full matrix is negative for all values of $r_2 \in (0,1).$

□

B. Index Theory.

Simsek, Ozdaglar, and Acemoglu (2007) present an extension of the Poincaré-Hopf index Theorem, which includes a uniqueness condition for continuous games as one of its applications. Their condition, which is also an extension of the Gale and Nikaido (1965) result discussed above, is implied by $(\boldsymbol{I}_n - \boldsymbol{J}_n(\boldsymbol{x}))$ being a *P* matrix, but is slightly more general. Also, their condition only needs to apply locally at the equilibrium and not globally. As with the Gale and Nikaido and Rosen results, however, the Simsek, Ozdaglar, and Acemoglu result demonstrates uniqueness within a bounded region and so, again, Theorem 6 provides an extension of the result into the case of unbounded strategy spaces.

C. Cournot Games.

There are many papers giving conditions for uniqueness in a Cournot quantity-setting game. These include Szidarovszky and Yakowitz (1977), Kolstad and Matheisen (1987), Gaudet and Salant (1991), and Long and Soubeyran (2000). All three of these papers provide conditions which imply that the best-response functions of players are negatively sloped, along with other conditions required to bound the set of prices over which demand is positive. As shown by the following, theorem, by using the general uniqueness theorem in this paper one only needs to require non-positively-sloped best-response functions; the bounding conditions are not necessary.

Theorem 8:

Let $\boldsymbol{A}_n$ be a square matrix with $a_{ii} = 1 \; \forall i$ and

$$a_{ij} = \alpha_i \in (-1, 0] \;\; \forall j \neq i, \;\; \forall i.$$

Then $|\boldsymbol{A}_n| > 0.$

Proof:

Given in the Appendix.

□

D. Row-Sum Conditions:

Cachon and Netessine (2004) show that a sufficient condition for the function, $\boldsymbol{f}$, to exhibit a contraction mapping is that, for all $\boldsymbol{x}$,

$$\sum_j \left| \frac{\partial f_i(\boldsymbol{x})}{\partial x_j} \right| < \beta \leq 1 \;\; \forall i \quad \text{or} \quad \sum_i \left| \frac{\partial f_j(\boldsymbol{x})}{\partial x_i} \right| < \beta \leq 1 \;\; \forall j.$$

That is, $\boldsymbol{f}$ exhibits a contraction mapping if the sum of the absolute values of the off-diagonal elements in the Jacobean matrix is bounded below one in each row or in each column. This result is established by showing that a function has a contraction mapping if the largest eigenvalue of the Jacobean matrix, $\boldsymbol{J}(\boldsymbol{x})$ is less than one, and that, using a result of Horn and Johnson (1996), this will hold if the maximum row sum or the maximum column sum is less than one. Although this approach is very different from ours, it is easy to show that this condition meets our requirement for there to be an iterative quasi-contraction mapping. Indeed, we can generalise the result a bit:

Theorem 9:

Let $\boldsymbol{A}_n$ be a square matrix with $a_{ii} \geq 0 \; \forall i$. If $\boldsymbol{A}_n$ has a dominant diagonal in the sense that there exist positive numbers, $d_1, d_2, \dots d_n$, such that either

$$\sum_{j \neq i} d_j \left| a_{ij} \right| < d_i a_{ii}, \text{ or}$$

$$\sum_{i \neq j} d_i \left| a_{ij} \right| < d_j a_{jj},$$

then $\left| \boldsymbol{A}_m \right| > 0 \ \forall m \leq n.$

Proof:

Given in the Appendix.

□

In the case where the function the function, $\boldsymbol{f}$, describes reaction functions so that $\partial f_i(\boldsymbol{x}) / \partial x_i = 0, \ \forall \boldsymbol{x}, \forall i,$ it is easy to show that the conditions for the Cachon and Netessine result imply that $\boldsymbol{A}_n$ is dominant diagonal with $d_i = 1 \ \forall i.$ Theorem 9 then generalises the contraction-mapping derived existence and uniqueness conditions in two ways. First, the theorem allows the rows or columns to be scaled by non-unitary d_i. Second, in the general case where it is not necessarily the case that $\partial f_i(\boldsymbol{x}) / \partial x_i = 0,$ it is easy to generate examples with $\partial f_i(\boldsymbol{x}) / \partial x_i < 0,$ under which, even with $d_i = 1 \ \forall i,$ $\boldsymbol{A}_n(\boldsymbol{x})$ is dominant diagonal but the maximum row sum and maximum column sum of $\boldsymbol{J}(\boldsymbol{x})$ exceeds one.

7. Conclusion.

This paper has presented a simple existence and uniqueness condition for continuous games which is both quite general and easy to apply. The condition encompasses and generalises a number of existing uniqueness conditions that were derived using a wide variety of approaches. The condition in this paper, then, provides a unifying framework for presenting those conditions. Most importantly, however, it extends those existing results to the case of games with unbounded strategy spaces.

As shown by the relative simplicity of the proofs of Theorems 8 and 9, the general condition—that the leading principal minors of the matrix $\boldsymbol{I}_n - \boldsymbol{J}_n(\boldsymbol{x})$ all be positive—lends itself to reasonably simple induction proofs for demonstrating that the condition holds in particular models. The result therefore has the potential to serve as a source for further existence and uniqueness conditions in specific games.

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Appendix

A. Proof of Theorem 8.

For ease of exposition, it will be convenient to prove a trivially generalised statement of Theorem 8 in which the diagonal elements of $\boldsymbol{A}_n$ can take any positive values and there can be a single row in which the common-off-diagonal elements take the same value as the diagonal element:

Theorem 8a:

Let Ω_n be the set of *n*x*n* matrices, $\boldsymbol{A}_n$, satisfying the following properties:

a) $a_{ii} => 0 \ \ \forall i$

b) $a_{ij} = b_i \in [0, a_{ii}] \quad \forall i, i \neq j.$

c) If $b_i = a_{ii}$ for some i, then $b_i < a_{ii} \ \ \forall j \neq i.$

Then

$$\left|\boldsymbol{A}_n\right| > 0 \qquad \forall \boldsymbol{A}_n \in \Omega_n, \forall n.$$

Proof:

The proof is by induction. The proposition is clearly true for n=1 and n=2. Now assume that there is some $\bar{n} > 2$ such that the proposition holds for all $n < \bar{n}.$ we will show that it then holds for $n = \bar{n}.$

Let $\boldsymbol{A}_n^{-i}$ be the submatrix obtained by removing the i'th row and column from $\boldsymbol{A}_n$. First note that if $b_i = 0$ for any i, then $\left|\boldsymbol{A}_n\right| = \left|\boldsymbol{A}_n^{-i}\right|$, and, since $\boldsymbol{A}_n \in \Omega_n \Rightarrow \boldsymbol{A}_n^{-i} \in \Omega_{n-1}$, the result holds by the induction assumption. We shall therefore only consider the case where $b_i > 0 \ \forall i.$

The proof follows by considering the matrix derived from $\boldsymbol{A}_n$ by replacing the diagonal terms in one row with the off-diagonal term for that row. We show that this change unambiguously reduces the determinant of the matrix, but results in a matrix with a non-negative determinant.

Formally, define the matrix $\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)$, which is derived from some *n*x*n* matrix, $\boldsymbol{A}_n \in \Omega_n$, by replacing a_{mm} with b_m. The determinant of this matrix is

$$\begin{aligned} & \left|\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)\right| = \left|\boldsymbol{A}_n\right| - (a_{mm} - b_m)\left|\boldsymbol{A}_n^{-m}\right| \\ \Rightarrow \quad & \left|\boldsymbol{A}_n\right| = \left|\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)\right| + (a_{mm} - b_m)\left|\boldsymbol{A}_n^{-m}\right|. \end{aligned} \tag{16}$$

Define $\bar{\Omega}_n \subset \Omega_n$ as the subset of matrices in Ω_n in which there is one row i where $b_i = a_{ii}$. Now consider some matrix $\boldsymbol{A}_n \in \bar{\Omega}_n$, and the adjusted matrix, $\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)$ where $b_m \neq a_{mm}$. $\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)$ is a matrix that has two rows in which all elements in the row are the same, and hence $\left|\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)\right| = 0$. Furthermore, $(a_{mm} - b_m) > 0$ and, since $\boldsymbol{A}_n \in \Omega_n \Rightarrow \boldsymbol{A}_n^{-m} \in \Omega_{n-1}$, by the induction assumption, $\left|\boldsymbol{A}_n^{-m}\right| > 0$. From Equation (16), then, we have $\left|\boldsymbol{A}_n\right| > 0 \; \forall \boldsymbol{A}_n \in \bar{\Omega}_n$.

Now consider a matrix, $\boldsymbol{A}_n \in \Omega_n \setminus \bar{\Omega}_n$ —that is a matrix for which $b_i < a_{ii} \; \forall i$ —and the derived matrix $\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)$ for some *m*. Again, we know that $(a_{mm} - b_m) > 0$ and, by the induction assumption, $\left|\boldsymbol{A}_n^{-m}\right| > 0$. Furthermore, $\boldsymbol{B}_m(\boldsymbol{A}_n, b_m) \in \bar{\Omega}_n$ and so, by the result shown in the previous paragraph, $\left|\boldsymbol{B}_m(\boldsymbol{A}_n, b_m)\right| > 0$. From Equation (16), then, we have $\left|\boldsymbol{A}_n\right| > 0 \; \forall \boldsymbol{A}_n \in \Omega_n \setminus \bar{\Omega}_n$, which establishes the result.

□

B. Proof of Theorem 9.

Theorem 4.C.1 in Takayama (1985), shows that a dominant diagonal matrix with no constraint on the sign of the diagonal elements must be non-singular. It is then straightforward to show that if the diagonal elements are all positive, the determinant must be positive. The proof is by induction.

Trivially, the 1x1 matrix whose single element is positive has a positive determinant. Now assume that the theorem holds for all matrices of size *m*-1, and let $\boldsymbol{A}_m$ be a dominant diagonal matrix. This implies that

$$\frac{\partial}{\partial a_{mm}}\left|\boldsymbol{A}_m\right| = \left|\boldsymbol{A}_{m-1}\right| > 0. \tag{17}$$

Now let

$$\hat{a}_{mm} = a_{mm} - \frac{\left|\boldsymbol{A}_m\right|}{\left|\boldsymbol{A}_{m-1}\right|}, \tag{18}$$

and let $\hat{\boldsymbol{A}}_m$ be the matrix created by replacing a_{mm} with $\hat{a}_{mm}$. Equations (17) and (18) then imply that

$$\left|\hat{\boldsymbol{A}}_m\right| = 0,$$

which from Takayama's result implies that $\hat{\boldsymbol{A}}_m$ cannot be dominant diagonal and hence that

$$\hat{a}_{mm} < a_{mm}. \tag{19}$$

Since $\left|\boldsymbol{A}_{m-1}\right| > 0$, (18) and (19) together imply that $\left|\boldsymbol{A}_m\right| > 0$.

□